

Minimalist approach to the classification of symmetry protected topological phases

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Abstract

A number of proposals with differing predictions (e.g. Borel group cohomology, oriented cobordism, group supercohomology, spin cobordism, etc.) have been made for the classification of symmetry protected topological (SPT) phases. Here we treat various proposals on an equal footing and present rigorous, general results that are independent of which proposal is correct. We do so by formulating a minimalist Generalized Cohomology Hypothesis, which is satisfied by existing proposals and captures essential aspects of SPT classification. From this Hypothesis alone, formulas relating classifications in different dimensions and/or protected by different symmetry groups are derived. Our formalism is expected to work for fermionic as well as bosonic phases, Floquet as well as stationary phases, and spatial as well as on-site symmetries.

Keywords: symmetry protected topological phases, generalized cohomology theories

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1. Introduction

The quest for a complete understanding of phases of matter has been a driving force in condensed matter physics. From the Landau-Ginzburg-Wilson paradigm [1] to topological insulators and superconductors [2–6] to topological orders [7] to symmetry protected topological (SPT) phases [8] to symmetry enriched topological phases [9–12], we have witnessed an infusion of ideas from topology into this century-old field. SPT phases are a relatively simple class of non-symmetry-breaking, gapped quantum phases and have been a subject of intense investigation in recent years [13]. As an interacting generalization of topological insulators and superconductors and intimate partner of topological orders [14], they exhibit such exotic properties as the existence of gapless edge modes, and harbor broad applications. They have also been increasingly integrated into other novel concepts such as many-body localization and Floquet phases [15–29].

Despite tremendous progress [30–53], a complete classification of SPT phases remains elusive. This is especially true when fermions, high (e.g. ≥ 3) spatial dimensions, or continuous symmetry groups are involved. A number of proposals have been made for the general classification of SPT phases: the Borel group cohomology proposal [33], the oriented cobordism proposal [35], Freed’s proposal [38, 39], and Kitaev’s proposal [40, 42] in the bosonic case; and the group supercohomology proposal [34], the spin cobordism proposal [36], Freed’s proposal [38, 39], and Kitaev’s proposal [42, 43] in the fermionic case. These proposals give differing predictions in certain dimensions for certain symmetry groups, and while more careful analysis [45–53] has uncovered previously overlooked phases and brought us closer than ever to our destination, we believe that we can do much more.

In this paper, we will take a novel, minimalist approach to the classification problem of SPT phases, by appealing to the following principle of Mark Twain’s [54]:

Distance lends enchantment to the view.

In this spirit, we will not commit ourselves to any particular construction of SPT phases, specialize to specific dimensions or symmetry groups, or investigate the completeness of any of the proposals above. Instead, we will put various proposals under one umbrella and present results that are independent of which proposal is correct. This will begin with the formulation of a hypothesis, we dub the Generalized Cohomology Hypothesis, that encapsulates essential attributes of SPT classification. These attributes will be shown to be possessed by various existing proposals and argued, on physical grounds, to be possessed by the unknown complete classification should it differ from existing ones. The results we present will be rigorously derived from this Hypothesis alone. Because we are taking a “meta” approach, we will not be able to produce the exact classification in a given dimension protected by a given symmetry group. We will be able, however, to *relate* classifications in different dimensions and/or protected by different symmetry groups. Such relations will be interpreted physically – this may require additional physical input, which we will keep to a minimum and state explicitly. A major advantage of this formalism is the universality of our results, which, as we said, are not specific to any particular construction.

What will enable us to relate different dimensions and symmetry groups is ultimately the fact that the Hypothesis is a statement about all dimensions and all symmetry groups simultaneously. Furthermore, due to a certain “symmetry” the Hypothesis carries, the relations we derive will hold in arbitrarily high dimensions. Finally, the Hypothesis is supposed to apply to fermionic phases as well as bosonic phases. Thus our formalism is not only independent of construction, but also independent of physical dimension and particle content, that is, bosons vs. fermions.

More specifically, the Hypothesis will be based on a prototype offered by Kitaev [40, 42, 43]. We will add a couple of new ingredients (additivity and functoriality; see below) and formulate the ideas in a language amenable to rigorous treatment. While the Hypothesis is informed by Refs. [40, 42, 43], our philosophy is fundamentally different. The goal of Refs. [40, 42, 43] was to classify SPT phases in ≤ 3 dimensions by incorporating into the Hypothesis current understanding of the classification of invertible topological orders. The goal of this paper is to make rigorous, maximally general statements about the classification of SPT phases by refraining from incorporating such additional data. The approach of Refs. [40, 42, 43] was concrete, whereas ours is minimalist.

Here is a preview of some of the fruits of this minimalist undertaking.

- (i) We will be able to relate the original definition of SPT phases [8, 31] to the one currently being developed by Refs. [35, 38, 39, 41, 42, 55], which is in terms of invertibility of phases and uniqueness of ground state on arbitrary spatial slices. According to the latter definition, the classification of SPT phases can be nontrivial even without symmetry. (For instance, the integer quantum Hall state represents an SPT phase in that sense.) We will show that SPT phases in the old sense are not only a subset, but in fact a direct summand¹, of SPT phases in the new sense. More precisely,

$$\left\{ \begin{array}{l} d\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \\ \text{in the new sense} \end{array} \right\} \cong \left\{ \begin{array}{l} d\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \\ \text{in the old sense} \end{array} \right\} \oplus \left\{ \begin{array}{l} d\text{-dimensional} \\ \text{invertible topo-} \\ \text{logical orders} \end{array} \right\}, \quad (1)$$

where invertible topological orders are synonymous with SPT phases (in the new sense) without symmetry, and d and G are arbitrary. We will also see the two definitions are nicely captured by two natural variants of a mathematical structure that we will introduce. These claims depend only on the Hypothesis, and are expounded upon in Sec. 6.1.

- (ii) We will be able to relate the classification of translationally invariant SPT phases to the classification of usual SPT phases. (From now on, SPT phases will mean SPT phases in the new sense.) The former are protected by a discrete spatial translational symmetry \mathbb{Z} as well as an internal symmetry G , whereas the latter are protected by G alone. It is conceivable that translational symmetry will refine the classification, but it is not clear whether every usual SPT phase will have a translationally invariant representative, whether every usual SPT phase will split into multiple phases, or whether all usual SPT phases will split into the same number of phases. To all three questions, we will give affirmative answers. More precisely, we will prove that there is a decomposition

$$\left\{ \begin{array}{l} d\text{-dimensional} \\ (\mathbb{Z} \times G)\text{-protected} \\ \text{SPT phases} \end{array} \right\} \cong \left\{ \begin{array}{l} (d-1)\text{-dimensional} \\ G\text{-protected SPT} \\ \text{phases} \end{array} \right\} \oplus \left\{ \begin{array}{l} d\text{-dimensional} \\ G\text{-protected} \\ \text{SPT phases} \end{array} \right\}, \quad (2)$$

such that forgetting the translational symmetry corresponds to projecting from the left-hand side onto the second direct summand in the right-hand side. These claims depend only on the Hypothesis and the belief that it applies to translational symmetries as well as internal symmetries for a suitable definition of translationally invariant SPT phases. These are the subject of Sec. 6.2.

- (iii) We will go on to argue, through a field-theoretic construction in App. B, that the inclusion of the first summand in the right-hand side into the left-hand side corresponds to a layering construction, where one produces a d -dimensional translationally invariant phase by stacking identical copies of a usual $(d-1)$ -dimensional phase.
- (iv) We will generalize the relation above to d -dimensional SPT phases protected by discrete translation in n directions. We will see a hierarchy of lower-dimensional classifications enter the decomposition, with $\binom{n}{k}$ direct summands in dimension $d-k$. (The relation above corresponds to $n=1$.) This is discussed in Sec. 6.3.
- (v) We will reinterpret the \mathbb{Z} above as discrete temporal translational symmetry. Accordingly, there will be a decomposition

$$\left\{ \begin{array}{l} d\text{-dimensional } G\text{-} \\ \text{protected Floquet} \\ \text{SPT phases} \end{array} \right\} \cong \left\{ \begin{array}{l} (d-1)\text{-dimensional } G\text{-} \\ \text{protected (stationary)} \\ \text{SPT phases} \end{array} \right\} \oplus \left\{ \begin{array}{l} d\text{-dimensional } G\text{-} \\ \text{protected (station-} \\ \text{ary) SPT phases} \end{array} \right\}. \quad (3)$$

We will give physical meaning to the projection maps onto the two direct summands in the right-hand side, in terms of pumping and Floquet eigenstates, respectively. What the relation tells us is

¹The direct sum is with respect to an abelian group structure of classification that we will describe later. Note that we could have used the direct product notation \times for groups, but the direct sum notation \oplus is more common for abelian groups in the mathematical literature.

that a d -dimensional Floquet SPT phase can pump any $(d - 1)$ -dimensional stationary SPT phase we want, that it can represent any d -dimensional (stationary) SPT phase we want, and that it is completely determined by these two pieces of information. Except for the pumping interpretation, these claims depend only on the Hypothesis and the belief that it applies to discrete temporal translational symmetry as well as internal symmetries for a suitable definition of Floquet SPT phases. These are discussed in Sec. 6.4.

- (vi) We will show that a similar decomposition exists for semidirect products $\mathbb{Z} \rtimes G$, and more generally $G_1 \rtimes G_2$, whose applications to space group-protected SPT phases will be discussed in Sec. 6.5.
- (vii) An enlargement of symmetry group can not only refine a classification but also eliminate certain phases, for a priori there may be obstructions to lifting an action of a smaller symmetry group over to a larger symmetry group. In Sec. 6.6, we will give a sufficient condition for the absence of such obstructions. More specifically, given $G' \subset G$, if one can find another subgroup $G'' \subset G$ such that $G'' \rtimes G' = G$, including the special case of direct product, then every G' -protected SPT phase will be representable by some G -protected SPT phase. This claim follows immediately from the Hypothesis.
- (viii) There are other results derived from the Hypothesis that we would rather defer to a subsequent paper due to our incomplete understanding. They are summarized in Sec. 7.

This paper is organized as follows. In Sec. 2, we establish conventions, define SPT phases, and comment on two elementary properties of SPT phases, additivity and functoriality, that will play a role in the Hypothesis. In Sec. 3, we introduce necessary mathematical concepts and formulate the Generalized Cohomology Hypothesis. In Sec. 4, we justify the Hypothesis on physical grounds. In Sec. 5, we present mathematical forms of the results we derived from the Hypothesis. In Sec. 6, we explore physical implications of these results. In Sec. 7, we summarize the paper, advertise further preliminary results, and suggest future directions.

A variety of topics are covered in the appendices. In App. A, we explain in more detail how existing proposals for the classification of SPT phases satisfy the Hypothesis. In App. B, we propose a field-theoretic construction to corroborate the weak-index interpretation in Sec. 6.2. In App. C, we present an equivalent but more succinct version of the Hypothesis using the terminology of category theory. In App. D, we explicitly show that the group cohomology construction [33] is additive and functorial. In App. E, we supply proofs to various lemmas and propositions in the paper. App. F is a review of notions in algebraic topology, category theory, and generalized cohomology theories.

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2. Generalities

2.1. Particle content, dimensionality, and symmetry action

Locality is defined differently for fermionic systems than for bosonic (i.e. spin) systems [56]. For this reason, classifications of bosonic phases and fermionic phases are traditionally done separately. While we will follow that tradition, our formalism works identically in the two cases. Therefore, we can omit the qualifiers “fermionic” and “bosonic” and simply speak of “SPT phases.”

By the dimension of a physical system, we always mean the spatial dimension. When it comes to mathematical construction, it is convenient to allow dimensions to be negative. If a purely mathematical result in this paper appears to contain a free variable d , then it should be understood that this result is

valid for all $d \in \mathbb{Z}$. If a physical result appears to contain a free variable d , then it should be understood that this result is valid for all $d \in \mathbb{Z}$ for which all dimensions involved are non-negative.

For simplicity, we assume all symmetry actions to be linear unitary. A generalization to antilinear antiunitary actions is possible (see Sec. 7) but beyond the purview of this paper.

We allow all topological groups satisfying the basic technical conditions in App. F.3 to be symmetry groups. Thus, a symmetry group can be finite or infinite, and discrete or non-discrete (also called “continuous”). In the non-discrete case, one must define what it means for a symmetry group G to act on a Hilbert space \mathcal{H} , that is whether we want a representation $\rho : G \rightarrow U(\mathcal{H})$ to be continuous, measurable², or something else, where $U(\mathcal{H})$ denotes the space of unitary operators on \mathcal{H} [33]. Conceivably, the Hypothesis can hold for one definition but fail for another, so some care is needed.

It is possible that the validity of the Hypothesis requires further restrictions on symmetry groups and symmetry actions, such as compactness and on-siteness, but there is a growing body of evidence [15–17, 30, 31, 57–65] against the necessity of such restrictions. It appears that discrete temporal translation [15–17], discrete spatial translation [30, 31], and other space group actions [57–65] may well fit into the same framework as on-site symmetry actions. In particular, Refs. [64, 65] maintained that the classification of d -dimensional G -protected topological phases is the same whether G is spatial or internal, provided that orientation-reversing symmetry operations (e.g. parity) are treated antiunitarily. In any case, on-site actions by finite groups are in the safe zone. We emphasize that the derivation of the mathematical results in Sec. 5 from the Hypothesis is independent of these considerations.

2.2. Mathematical notation and conventions

We denote bijections and homeomorphisms by \approx , isomorphisms of algebraic structures by \cong , homotopy or pointed homotopy by \sim , and homotopy equivalences or pointed homotopy equivalences by \simeq . We denote the one-point set, the unit interval (i.e. $[0, 1]$), the boundary of the unit interval (i.e. $\{0, 1\}$), the n -sphere, the n -disk, and the boundary of the n -disk by pt , I , ∂I , \mathbf{S}^n , \mathbf{D}^n , and $\partial \mathbf{D}^n$, respectively.

Unless stated otherwise, “map” always means continuous map, “group” always means topological group, and “homomorphism” between groups always means continuous homomorphism. For experts, the technical conventions in App. F.3 are observed throughout, except in Apps. F.1–F.3.

2.3. Definition of SPT phases

2.3.1. Old definition of SPT phases

Traditionally, the definition of SPT phases goes as follows [8, 31]. First, one defines a trivial system to be a local, gapped system whose unique ground state is a product state. Then, one defines a short-range entangled (SRE) system to be a local, gapped³ system that can be deformed to a trivial one via local, gapped systems. Finally, one defines a G -protected SPT phase to be an equivalence class of G -symmetric, non-symmetry-breaking⁴ SRE systems with respect to the following equivalence relation: two such systems are equivalent if they can be deformed into each other via G -symmetric, non-symmetry-breaking SRE systems.

2.3.2. New definition of SPT phases

Explicit as the definition above is, we shall adopt a different definition that will turn out to be extremely convenient for our formalism, at the expense of including more phases. The set of SPT phases in the old sense will be shown to sit elegantly inside the set of SPT phases in the new sense, undisturbed, and they can be readily recovered. The definition spelled out below is based on the ideas in Refs. [35, 38, 39, 41, 42, 55].

To begin, let us assume that the terms “system,” “local,” “gapped,” “ G -symmetric,” “non-symmetry-breaking,” and “deformation” have been defined. Given two arbitrary systems a and b of the same

²The measurability of $(d+1)$ -cochains as postulated in Ref. [33] reduces to the measurability of ρ when $d = 0$.

³We do not consider a system with accidental degeneracy in the thermodynamic limit to be gapped.

⁴Note that “ G -symmetric” is an adjective qualifying Hamiltonians while “non-symmetry-breaking” is an adjective qualifying ground states.

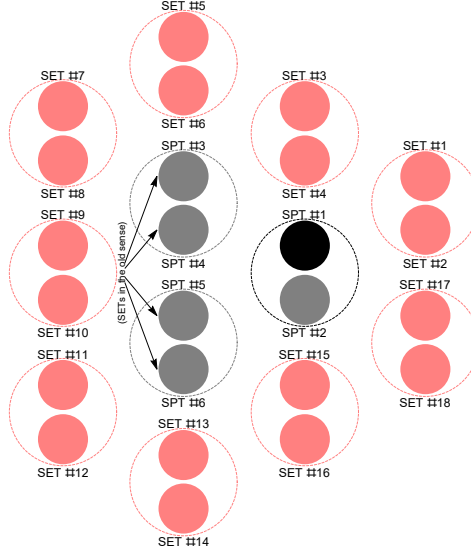


Figure 1: (color online). Schematic illustration of the structure of the space of d -dimensional, G -symmetric, non-symmetry-breaking, local, gapped systems. Each deformation class, shown as a patch here, is called a G -protected topological phase. Each invertible (respectively non-invertible) class, shown as a gray or black (respectively pink) patch, is called an SPT (respectively SET) phase. The identity class, shown as a black patch, is called the trivial SPT phase. Dashed circles are meant to indicate, by forgetting the symmetry, that more systems will be allowed and that distinct phases can become one.

dimension, we write $a + b$ (no commutativity implied; this is just a notation) for the composite system formed by stacking b on top of a . However the aforementioned terms may be defined, it seems reasonable to demand the following:

- (i) $a + b$ is well-defined.
- (ii) If both a and b are local, gapped, G -symmetric, or non-symmetry-breaking, then $a + b$ is also local, gapped, G -symmetric, or non-symmetry-breaking, respectively.
- (iii) A deformation of either a or b also constitutes a legitimate deformation of $a + b$.

We will speak of deformation class, which, as usual, is an equivalence class of systems with respect to the equivalence relation defined by deformation (possibly subject to constraints, as discussed in the next paragraph)⁵.

Now, let G be a symmetry group and d be a non-negative integer. Consider the set $\mathcal{M}^d(G)$ of deformation classes of d -dimensional, local, gapped, G -symmetric, non-symmetry-breaking systems. We have seen that there is a binary operation on the set of such systems, given by stacking, which descends to a binary operation on $\mathcal{M}^d(G)$, owing to property (iii). We define the *trivial d -dimensional G -protected SPT phase* to be the identity of $\mathcal{M}^d(G)$ with respect to the said binary operation. We define a *d -dimensional G -protected SPT phase* to be an invertible element of $\mathcal{M}^d(G)$. We define a *d -dimensional G -protected symmetry enriched topological (SET) phase* to be a non-invertible element of $\mathcal{M}^d(G)$. In general, we call an element of $\mathcal{M}^d(G)$ a *d -dimensional G -protected topological phase*. An illustration of these concepts appears in Fig. 1.

In mathematical jargon, SPT phases are thus the group of invertible elements of the monoid $\mathcal{M}^d(G)$ of d -dimensional G -protected topological phases. We will see later that $\mathcal{M}^d(G)$ is commutative. This means that the d -dimensional G -protected SPT phases form not just a group, but an abelian group. This is elaborated upon in Sec. 2.4.1.

⁵If a deformation is defined to be a path in a space of systems that comes with a topology, then a deformation class is nothing but a path component of the space.

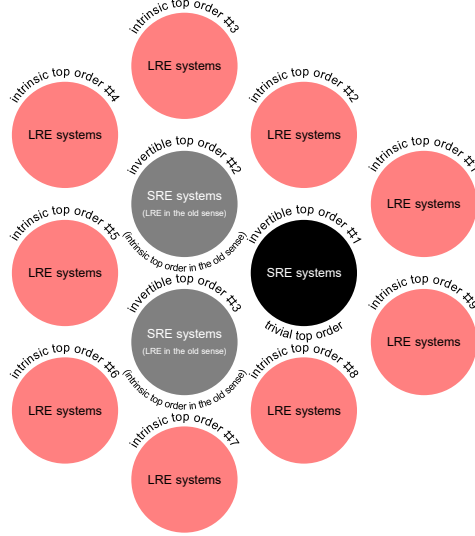


Figure 2: (color online). Schematic illustration of the structure of the space of d -dimensional local, gapped systems. Each deformation class, shown as a patch here, is called a topological order. Each invertible (respectively non-invertible) class, shown as a gray or black (respectively pink) patch, is called an invertible (respectively intrinsic) topological order. The identity class, shown as a black patch, is called the trivial topological order, which is in particular invertible. A system is called SRE (respectively LRE) if it belongs to an invertible (respectively intrinsic) topological order.

Note that we have made no mention of SRE systems so far. Instead, SPT and SET phases naturally fall out of the binary operation given by stacking. The uniqueness of identity and inverses and the abelian group structure of SPT phases come about for free. This is in line with the minimalism we are after and is we think the beauty of the definition.

Let us introduce special names for the special case of trivial symmetry group $G = 0$. The trivial SPT phase in this case can be called the *trivial topological order*; an SPT phase, an *invertible topological order*; an SET phase, an *intrinsic topological order*; and any element of $\mathcal{M}^d(0)$, a *topological order*. We may call a system *short-range entangled (SRE)* if it represents an invertible topological order, and *long-range entangled (LRE)* otherwise. An illustration of these concepts appears in Fig. 2.

2.3.3. Comparison between old and new definitions of SPT phases

To make contact with the old definition of SPT phases [8, 31], we note that all trivial systems in the old sense represent the identity element of $\mathcal{M}^d(0)$, where 0 denotes the trivial group. Hence, SRE systems in the old sense are precisely those SRE systems in our sense that happen to lie in this identity class. Similarly, SPT phases in the old sense are precisely those SPT phases in our sense that, by forgetting the symmetry, represent the said identity class. This shows that the SPT phases in the old sense are a subset of the SPT phases in our sense. One of our results in this paper is that the former form a subgroup, in fact a direct summand, of the latter. These are illustrated in Figs. 1 and 2.

What is also clear is that the classification of SPT phases (according to our definition; same below) can be nontrivial even for the trivial symmetry group. This amounts to saying that there can exist nontrivial invertible topological orders, or that the set of SRE systems are partitioned into more than one deformation classes in the absence of symmetry. Examples of systems that represent nontrivial invertible topological orders are given in Table 1. While this may seem to contradict the original idea [8] of symmetry protection, it is the new notion of short-range entanglement not the old one that is closely related and potentially equivalent to the condition of unique ground state on spatial slices of arbitrary topology, and in two dimensions, the condition of no nontrivial anyonic excitations [13, 35, 38, 39, 41, 42, 55], both of which are more readily verifiable, numerically and experimentally, than the deformability to product states.

Table 1: Examples of systems that represent nontrivial invertible topological orders [42]. They are legitimate representatives of SPT phases according to our definition but fall outside the realm of Refs. [8, 31].

Particle content	Dimension	System
Fermion	0	An odd number of fermions
Fermion	1	The Majorana chain [66]
Fermion	2	$(p + ip)$ -superconductors [67–69]
Boson	2	The E_8 -model [45, 70, 71]

2.4. Elementary properties of SPT phases

In this subsection, we discuss two elementary properties of the classification of SPT phases that will play a role in the Hypothesis. These follow essentially from the definition and should be features of any classification proposal.

2.4.1. Additivity

Additivity says that the d -dimensional G -protected SPT phases form a discrete⁶ abelian group with respect to stacking. To see this, we first note that stacking of d -dimensional G -protected topological phases is tautologically associative (Fig. 3). We then note that any G -symmetric system with a product state as the unique gapped ground state, which always exists, represents an identity with respect to stacking. Since SPT phases are invertible by definition, a discrete group structure is defined.

This leaves commutativity. We recall, in order to compare systems defined on different Hilbert spaces, that one would usually allow for “embedding” of smaller Hilbert spaces into larger Hilbert spaces⁷. This is known as an isometry [30, 31, 72, 73]. Given two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 – these are supposed to be associated to individual sites of two different systems – the Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}_2 \otimes \mathcal{H}_1$ are isomorphic. Embedding them into $(\mathcal{H}_1 \otimes \mathcal{H}_2) \oplus (\mathcal{H}_2 \otimes \mathcal{H}_1) \cong \mathbb{C}^2 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$, we can then interpolate between the two in a canonical, symmetry-preserving fashion. Therefore, the resulting phase is independent of the order of stacking.

Note that the above also shows that the d -dimensional G -protected topological phases form a discrete commutative monoid $\mathcal{M}^d(G)$.

(Some definitions of SPT phases admit the coexistence of multiple trivial phases [61, 74], but this can always be salvaged by declaring the identity under stacking to be the true trivial phase, which is unique by elementary group theory.)

2.4.2. Functoriality

Functoriality says that every homomorphism $\varphi : G' \rightarrow G$ between any symmetry groups G' and G induces a homomorphism φ^* from the discrete abelian group of d -dimensional G -protected SPT phases to the discrete abelian group of d -dimensional G' -protected SPT phases. Note that the direction of mapping is reversed. Implicit here is the assumption that the coherence relation $(\varphi \circ \phi)^* = \phi^* \circ \varphi^*$ be satisfied for all composable homomorphisms φ and ϕ .

Let us first understand this in the special case where G' is a subgroup of G and φ is the inclusion. A d -dimensional G -protected SPT phase is represented by a d -dimensional, local, gapped, G -symmetric, non-symmetry-breaking system. By forgetting all symmetry operations outside the subgroup G' , we can view this same system as a representative of a d -dimensional G' -protected SPT phase. Since this applies to paths of systems as well, we get a well-defined map from the set of d -dimensional G -protected SPT phases to the set of d -dimensional G' -protected SPT phases. This is the induced map φ^* . It is easy to check that φ^* preserves discrete abelian group structure. Moreover, such maps can be composed. For instance, we can further forget G' entirely to obtain a map into the set of d -dimensional invertible

⁶Recall that “group” in this paper means “topological group.” This is why we need the adjective “discrete” here, as the abelian group of SPT phases is not endowed with a topology.

⁷More precisely, we want to “embed” representations of the symmetry group rather than Hilbert spaces.

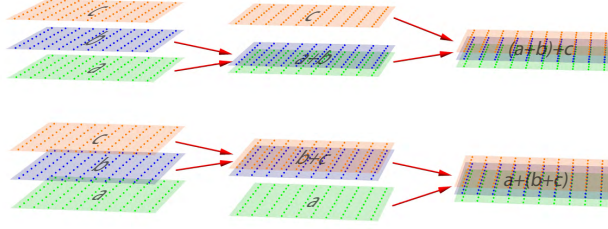


Figure 3: (color online). Stacking is associative. Given three systems, a (green), b (blue), and c (orange), combining a and b first and then c (upper panel) produces the same system as combining b and c first and then a (lower panel) does.

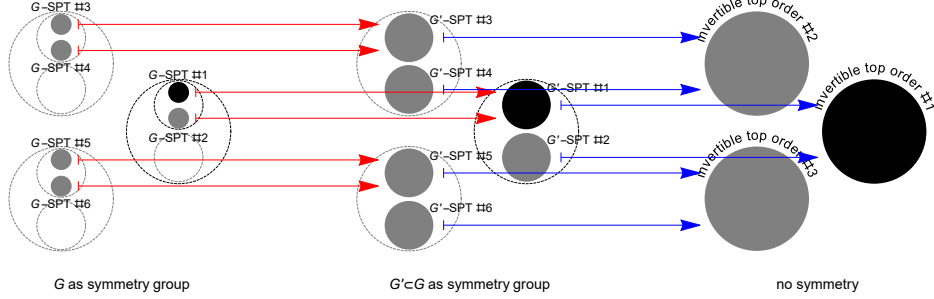


Figure 4: (color online). Given $G' \subset G$, a representative of a d -dimensional G -protected SPT phase can also be viewed as a representative of a d -dimensional G' -protected SPT phase, which in turn can be viewed as a representative of a d -dimensional invertible topological order, by forgetting first the symmetry operations outside G' and then G' itself. This defines a map from the set of d -dimensional G -protected SPT phases to the discrete abelian group of d -dimensional G' -protected SPT phases, and then to the set of d -dimensional invertible topological orders.

topological orders. Forgetting symmetry operations in two steps is clearly equivalent to forgetting them all at once, which is the origin of the coherence relation $(\varphi \circ \phi)^* = \phi^* \circ \varphi^*$. These are illustrated in Fig. 4.

The general case where $\varphi : G' \rightarrow G$ is an arbitrary homomorphism only requires a small modification. A d -dimensional G -protected SPT phase is represented by a triple $(\mathcal{H}, \rho, \hat{H})$, where \hat{H} is a Hamiltonian and $\rho : G \rightarrow U(\mathcal{H})$ is a representation of G on some Hilbert space \mathcal{H} . By precomposing φ , we obtain a representation $\rho \circ \varphi : G' \xrightarrow{\varphi} G \xrightarrow{\rho} U(\mathcal{H})$ of G' . Then the triple $(\mathcal{H}, \rho \circ \varphi, \hat{H})$ represents a d -dimensional G' -protected SPT phase. This defines the map φ^* .

Note that the same argument also shows that every homomorphism $\varphi : G' \rightarrow G$ between any symmetry groups G' and G induces a homomorphism $\varphi^* : \mathcal{M}^d(G) \rightarrow \mathcal{M}^d(G')$ between the monoids of d -dimensional G - and G' -protected topological phases.

3. The Generalized Cohomology Hypothesis

In this section, we will state the Generalized Cohomology Hypothesis, which is the foundation of our formalism. Intuitively, the Hypothesis says that the classifications of SPT phases in different dimensions protected by different symmetry groups are intertwined in some intricate fashion, so that all information can be encoded into what is called an Ω -spectrum. And just like proteins are produced from genes through the processes of transcription and translation, the classifications of d -dimensional G -protected SPT phases for varying d and G can be produced from the Ω -spectrum through the classifying space construction and homotopy theory.

An Ω -spectrum is by definition a sequence of pointed topological spaces F_d indexed by integers $d \in \mathbb{Z}$ together with pointed homotopy equivalences $F_d \simeq \Omega F_{d+1}$, where ΩF_{d+1} is the loop space of F_{d+1} (see App. F.1). As discussed in Sec. 4.6, F_d is believed to be the space of d -dimensional SRE states, and the pointed homotopy equivalences $F_d \simeq \Omega F_{d+1}$ can be given physical interpretations as well. Note that

shifting d turns an Ω -spectrum into another Ω -spectrum. This is responsible for the validity of the results in Secs. 5 and 6 in arbitrarily high dimensions.

Definition 3.1. An (unreduced) generalized cohomology theory h has an Ω -spectrum $(F_d)_{d \in \mathbb{Z}}$ as its data. Given an integer d , it assigns to each topological space X the discrete abelian group $h^d(X) := [X, F_d]$, i.e. the homotopy classes of maps from X to F_d .⁸

Definition 3.2. A reduced generalized cohomology theory \tilde{h} has an Ω -spectrum $(F_d)_{d \in \mathbb{Z}}$ as its data. Given an integer d , it assigns to each pointed topological space X the discrete abelian group $\tilde{h}^d(X) := \langle X, F_d \rangle$, i.e. the homotopy classes of pointed maps from X to F_d .

Different choices of Ω -spectrum can give wildly different generalized cohomology groups $h^d(X)$'s and $\tilde{h}^d(X)$'s. This is the degree of freedom that will allow us to encompass various inequivalent classification proposals. Furthermore, unreduced and reduced theories come hand in hand and can be recovered from each other.

The discrete abelian group structure on $h^d(X)$ is defined via the bijection $h^d(X) := [X, F_d] \approx [X, \Omega F_{d+1}]$. Given two classes $[c_1], [c_2] \in h^d(X)$ represented by maps $c_1, c_2 : X \rightarrow \Omega F_{d+1}$, we define $[c_1] + [c_2]$ by concatenating the loops $c_1(x)$ and $c_2(x)$ for each x . Further replacing ΩF_{d+1} by $\Omega^2 F_{d+2}$, one can show that $[c_1] + [c_2] = [c_2] + [c_1]$. The reduced case is similar.

h^d is also functorial, in that every map $f : X \rightarrow Y$ induces a homomorphism $f^* : h^d(Y) \rightarrow h^d(X)$ so that $(f \circ g)^* = g^* \circ f^*$ for all composable f and g . Given a class $[c] \in h^d(Y)$ represented by a map $c : Y \rightarrow F_d$, we define $f^*([c])$ by precomposing f with c . The reduced case is similar.

Before stating the Generalized Cohomology Hypothesis, we recall there is a so-called classifying space functor B (see App. F.4). It assigns a pointed topological space BG to each group G , and a pointed map $B\varphi : BG' \rightarrow BG$ to each homomorphism $\varphi : G' \rightarrow G$. As a result, the composition $h^d(B-)$ of B and h^d assigns a discrete abelian group $h^d(BG)$ to each group G , and a homomorphism $\varphi^* : h^d(BG) \rightarrow h^d(BG')$ to each homomorphism $\varphi : G' \rightarrow G$. The reduced case is similar. We are now ready to state the

Generalized Cohomology Hypothesis. There exists an (unreduced) generalized cohomology theory h such that, given any dimension $d \geq 0$ and symmetry group G , $h^d(BG)$ classifies d -dimensional G -protected SPT phases (see Sec. 2.3.2), with its discrete abelian group structure corresponding to stacking (see Sec. 2.4.1) and its functorial structure corresponding to replacing symmetry groups (see Sec. 2.4.2).

4. Justification of the Hypothesis

Before taking off from the Hypothesis, we must explain how we arrived at it. We devote this section to the justification of the Hypothesis.

4.1. Additivity and functoriality

We have seen that every generalized cohomology theory is additive and functorial. This is encouraging, as additivity and functoriality are basic to the classification of SPT phases and should be features of any classification proposal.

⁸This differs from the standard definition in two ways, even when the Brown representability theorem is assumed: first, the representing Ω -spectrum is part of the data; and second, we are not considering pairs of spaces. These differences, however, are completely innocuous.

Table 2: Classic examples of generalized cohomology theories and spectra that represent them [75, 76]. Here, $K(A, n)$ denotes the n -th Eilenberg-Mac Lane space of A (see App. F.4), and U denotes the infinite unitary group $U(\infty) = \bigcup_{i=1}^{\infty} U(i)$.

Theory	Spectrum	Standard notation	Explicit expression
Ordinary cohomology theory with coefficient group A	Eilenberg-Mac Lane spectrum of A	HA or $H^\bullet(-; A)$	$F_n = \begin{cases} K(A, n), & n \geq 0 \\ \text{pt}, & n < 0 \end{cases}$
Real K -theory	Real K -theory spectrum	KO	Periodic: $F_n \simeq F_{n+8}$
Complex K -theory	Complex K -theory spectrum	KU	$F_n = \begin{cases} \mathbb{Z} \times BU, & n \text{ even} \\ U, & n \text{ odd} \end{cases}$
Stable cohomotopy	Sphere spectrum	S	$F_n = \varinjlim \Omega^m \mathbf{S}^{n+m}$
Oriented cobordism	Thom spectrum of SO	MSO	$F_n = \varinjlim \Omega^m MSO_{n+m}$
Unoriented cobordism	Thom spectrum of O	MO	$F_n = \varinjlim \Omega^m MO_{n+m}$
Spin cobordism	Thom spectrum of $Spin$	$MSpin$	$F_n = \varinjlim \Omega^m MSpin_{n+m}$
Pin^\pm cobordism	Thom spectrum of Pin^\pm	$MPin^\pm$	$F_n = \varinjlim \Omega^m MPin_{n+m}^\pm$

4.2. Ubiquity of generalized cohomology theories

To give a feeling of the ubiquity of generalized cohomology theories, we have listed some classic examples in Table 2. Note that the first entry already hosts infinitely many possibilities, corresponding to different A 's. Other entries have obvious generalizations to other structure groups. Furthermore, one can synthesize new generalized cohomology theories from old ones in at least two ways. The first way is to take products of Ω -spectra degree-wise: given $(F'_d)_{d \in \mathbb{Z}}$ and $(F''_d)_{d \in \mathbb{Z}}$, we define

$$F_d := F'_d \times F''_d, \quad (4)$$

so that

$$[X, F_d] \cong [X, F'_d] \oplus [X, F''_d], \quad (5)$$

$$\langle X, F_d \rangle \cong \langle X, F'_d \rangle \oplus \langle X, F''_d \rangle. \quad (6)$$

The second way is to take the smash product of the corresponding CW-spectra of two given Ω -spectra [75].

It therefore would not be surprising if SPT phases were classified by a generalized cohomology theory of some sort. Better yet, the above operations on spectra could allow one to improve approximate classifications upon, for instance, the discovery of a class of systems exhibiting new physical effects⁹.

4.3. Existing proposals as special cases

One of the main motivations for the Hypothesis is the fact that it is satisfied by various existing proposals for the classification of SPT phases [40, 42, 43]. These include the Borel group cohomology proposal [33], the oriented cobordism proposal [35], and Kitaev's proposal [40, 42] in the bosonic case; and the group supercohomology proposal [34], the spin cobordism proposal [36], and Kitaev's proposal [42, 43] in the fermionic case. (Freed's proposals appear to be more nuanced; see the original papers [38, 39].) Their spectra are summarized in Table 3. We have checked, for finite symmetry groups, that the additive and functorial structures of the Borel group cohomology proposal indeed correspond to stacking phases and replacing symmetry groups, respectively; see App. D. The same can only be done to a lesser extent for non-finite groups or for the other proposals, where explicit lattice models are unavailable. Exactly how these proposals fit into our framework will be expounded upon in App. A.

The first entry in Table 3 may look odd, since Borel group cohomology as defined in Ref. [33] is an algebraic structure not a topological one. The equivalence between the two relies on the well-known natural isomorphism $H_{\text{group}}^{d+1}(G; A) \cong H^{d+1}(BG; A)$ for any coefficient A and discrete group G [77]. See App. A.1 for detail.

⁹We thank Christian Schmid for suggesting this.

Table 3: Generalized cohomology theories that have been proposed to classify SPT phases, and spectra that represent them. Here, we reused the notation in Table 2. $\mathbb{C}P^\infty = \bigcup_{i=1}^\infty \mathbb{C}P^i$ denotes the infinite projective space, and π_i and k_i denote the i -th homotopy group and the i -th k -invariant [78], respectively. The $\mathbb{C}P^\infty$ in F_0 , \mathbb{Z}_2 in fermionic F_0 , \mathbb{Z}_2 in fermionic F_1 , and \mathbb{Z} in bosonic F_2 have to do with Berry’s phase, fermion parity, the Majorana chain, and the E_8 -model, respectively (cf. Table 1) [40, 42, 43]. More details of these proposals can be found in App. A.

Classification proposal	Spectrum	Further information
Borel group cohomology as in Ref. [33]	Shifted $H\mathbb{Z}$	$F_d = \begin{cases} K(\mathbb{Z}, d+2), & d \geq -2, \\ \text{pt}, & d < -2. \end{cases}$ In particular, $F_0 \simeq \mathbb{C}P^\infty$.
Group supercohomology as in Ref. [34]	“Twisted product” of $H\mathbb{Z}_2$ and shifted $H\mathbb{Z}$	F_d can be constructed as a Postnikov tower: $\pi_i(F_d) \cong \begin{cases} \mathbb{Z}_2, & i = d, \\ \mathbb{Z}, & i = d+2, \quad k_{d+1} = \beta \circ Sq^2, \\ 0, & \text{otherwise}, \end{cases}$ where Sq^2 is the Steenrod square and β is the Bockstein homomorphism associated with $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$ [78]. In particular, $F_0 \simeq \mathbb{C}P^\infty \times \mathbb{Z}_2$ and $F_1 \simeq K(\mathbb{Z}, 3) \times K(\mathbb{Z}_2, 1)$.
Oriented cobordism as in Ref. [35] Spin cobordism as in Ref. [36] Kitaev’s bosonic proposal [40, 42]	Related to MSO Related to $MSpin$ Constructed from physical knowledge	See App. A. See App. A. F_d is uniquely determined in low dimensions: $F_d = \begin{cases} K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty, & d = 0, \\ K(\mathbb{Z}, 3), & d = 1, \\ K(\mathbb{Z}, 4) \times \mathbb{Z}, & d = 2, \\ K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 1) \simeq K(\mathbb{Z}, 5) \times \mathbf{S}^1, & d = 3. \end{cases}$
Kitaev’s fermionic proposal [42, 43]	Constructed from physical knowledge	See App. A. $F_0 = K(\mathbb{Z}, 2) \times \mathbb{Z}_2 \simeq \mathbb{C}P^\infty \times \mathbb{Z}_2$ is uniquely determined, and $F_{d>0}$ are partially determined. See App. A.

4.4. Rationale behind classifying spaces

The use of classifying spaces BG signifies a gauge-theory nature of SPT phases. More precisely, it suggests, for the purpose of classifying G -protected SPT phases, that it suffices to look at gauge theories with structure group G even though most systems are not gauge theories. An element $[c]$ of $h^d(BG)$ can be thought of as a generalized topological term, or more precisely, a characteristic class [79].

Let us elucidate this with a familiar example: the first Chern class c_1 , which assigns an element $c_1(\xi)$ of $H^2(X; \mathbb{Z})$ to each $U(1)$ -bundle ξ over X . In a physical context, X would be a Brillouin zone, ξ would be a family of Bloch wave functions over it, and $c_1(\xi)$ would be expressed as Berry’s curvature [80]. In general, however, X can vary, and c_1 has an important property called naturality: if ξ is a bundle over X , $f: Y \rightarrow X$ is a map, and $f^*\xi$ is the pull-back bundle over Y , then $c_1(f^*\xi) = f^*(c_1(\xi))$, where the f^* in the right-hand side is the induced homomorphism $f^*: H^2(Y; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$. Thus, the first Chern class of a pull-back bundle is determined by that of the original bundle. Since every $U(1)$ -bundle over any space X is the pull-back along some map $f: X \rightarrow BU(1)$ of the universal bundle $\xi_{U(1)}: EU(1) \rightarrow BU(1)$ over the classifying space $BU(1)$ (see App. F.4), we can regard c_1 simply as an element of $H^2(BU(1); \mathbb{Z})$.

In the general case, different gauge field configurations in a G -gauge theory over a manifold X can be thought of as different principal G -bundles over X . Reversing the logic in the previous paragraph, we see that a generalized cohomology class $[c] \in h^d(BG)$ would assign an element $[c](\xi)$ of $h^d(X)$ to each gauge field configuration ξ over X . Just like the first Chern class can be integrated over X to produce the first Chern number, the element $[c](\xi)$ can also sometimes be paired with the fundamental class of X to produce a characteristic number. The latter is supposed to be the topological action (or action amplitude) evaluated at the gauge field configuration ξ .

4.5. Lattice models for arbitrary generalized cohomology theory

Esoteric as generalized cohomology theories may seem, Refs. [42, 43] have actually outlined a way to construct lattice models given any such theory h . It can be thought of as a generalization of the group cohomology construction [33] where additional degrees of freedom are placed on the d -simplices of a d -dimensional system. The input is now a map $c: BG \rightarrow F_d$ instead of a $(d+1)$ -cocycle. More details of the construction can be found in App. B.1.

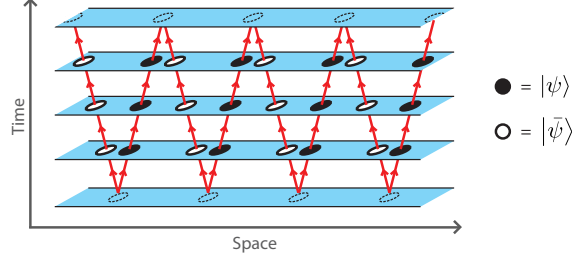


Figure 5: (color online). An illustration in the $d = 0$ case of the process that defines the map f in the pumping interpretation of Ω -spectrum, where only a finite segment of an infinite or periodic system is drawn.

4.6. Physical interpretations of Ω -spectrum

Refs. [40, 42, 43] proposed a physical interpretation of the spaces F_d in the Ω -spectrum: F_d is the space of d -dimensional SRE states (i.e. ground states of SRE systems), and the basepoint of F_d is a particular d -dimensional trivial SRE state (e.g. a product state). There are two interpretations of the pointed homotopy equivalences $F_d \simeq \Omega F_{d+1}$ as discussed below.

4.6.1. Dimension reduction interpretation

A pointed homotopy equivalence $F_d \simeq \Omega F_{d+1}$ consists of a pair of maps $f : F_d \hookrightarrow \Omega F_{d+1} : g$ such that $f \circ g \sim \text{id}$ and $g \circ f \sim \text{id}$. Recall that each element of ΩF_{d+1} is by definition a loop in F_{d+1} based at the basepoint of F_{d+1} . In the dimension reduction interpretation [43], g is defined by first interpreting a loop l in F_{d+1} as a pattern of $(d+1)$ -dimensional SRE states (with the endpoints of l corresponding to spatial infinity) and then taking the domain wall, where we note, because l is based at the basepoint of F_{d+1} , that the pattern is trivial far away from the domain wall. The spirit here is similar to that of the Jackiw-Rebbi soliton [81], for which the pattern is given by the spatially dependent mass term. The other map, f , is defined by inserting a d -dimensional SRE state $|\psi\rangle$ into a $(d+1)$ -dimensional trivial bulk and spreading it out in the normal direction, which then becomes a pattern of $(d+1)$ -dimensional SRE states and can be identified with an element of ΩF_{d+1} . Note that the dimension reduction interpretation is compatible with the identification of the discrete abelian group structure of $h^d(BG)$ with stacking. Namely, concatenating loops in F_{d+1} corresponds to stacking d -dimensional SRE states, and vice versa. That neither g nor f takes us out of the space of SRE (i.e. invertible-up-to-homotopy) states can be argued for by considering the reverse loop \bar{l} and the (up-to-homotopy) inverse $|\bar{\psi}\rangle$ of $|\psi\rangle$, respectively.

4.6.2. Pumping interpretation

In the pumping interpretation [40, 42], $g : \Omega F_{d+1} \rightarrow F_d$ is defined by interpreting a loop l in F_{d+1} as an adiabatic evolution and measuring the d -dimensional SRE state that is pumped across an imaginary cut. On the other hand, $f : F_d \rightarrow \Omega F_{d+1}$ is defined by assigning the following adiabatic evolution to a given a d -dimensional SRE state $|\psi\rangle$. Namely, we first create an alternating stack of $|\psi\rangle$ and $|\bar{\psi}\rangle$ and then annihilate them with neighbors, as depicted in Fig. 5. In the $d = 0$ case, this is reminiscent of the Chalker-Coddington model [82], although Ref. [82] was considering dynamics of real electrons not adiabatic evolution of SRE states. Note that the pumping interpretation is compatible with the identification of the discrete abelian group structure of $h^d(BG)$ with stacking. Namely, concatenating loops in F_{d+1} corresponds to stacking d -dimensional SRE states. That neither f nor g takes us out of the space of SRE states can be argued for by considering $|\bar{\psi}\rangle$ and the loop formed by concatenating l and \bar{l} , respectively.

5. Consequences of the Hypothesis: Mathematical Results

In this section, we discuss mathematical consequences of the Hypothesis. Their physical implications will be explored in Sec. 6. We stress that the results here depend on nothing beyond the Hypothesis. In fact, they are properties of all generalized cohomology theories.

In what follows, we will denote by $(F_d)_{d \in \mathbb{Z}}$ an arbitrary Ω -spectrum, and by h and \tilde{h} the unreduced and reduced generalized cohomology theories it defines, respectively.

5.1. Relationship between reduced and unreduced generalized cohomology theories

Lemma 5.1. Let G be any group and 0 be the trivial group. There is a natural split short exact sequence,

$$0 \longrightarrow \tilde{h}^d(BG) \xrightarrow{i} h^d(BG) \xleftarrow{s} \xrightarrow{p} h^d(B0) \longrightarrow 0 \quad (7)$$

with s induced by the epimorphism $G \twoheadrightarrow 0$, p induced by the monomorphism $0 \hookrightarrow G$, and i given by forgetting basepoints.

PROOF. See App. E. □

Corollary 5.2. Let G be any group and 0 be the trivial group. There is a natural isomorphism,

$$h^d(BG) \cong \tilde{h}^d(BG) \oplus h^d(B0). \quad (8)$$

□

5.2. A generalized Künneth formula for $\mathbb{Z} \times G$

Proposition 5.3. Let G be any group. There is a natural commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & h^{d-1}(BG) & \xrightarrow{\tilde{\alpha}} & \tilde{h}^d(B(\mathbb{Z} \times G)) & \xrightarrow{\tilde{\beta}} & \tilde{h}^d(BG) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & h^{d-1}(BG) & \xrightarrow{\alpha} & h^d(B(\mathbb{Z} \times G)) & \xrightarrow{\beta} & h^d(BG) \longrightarrow 0 \end{array} \quad (9)$$

where $\tilde{\beta}$ and β are induced by the monomorphism $G \hookrightarrow \mathbb{Z} \times G$, $g \mapsto (0, g)$, the two vertical maps are obtained by forgetting basepoints, $\tilde{\alpha}$ is the composition of the obvious maps

$$\begin{aligned} [BG, F_{d-1}] &\xrightarrow{\cong} [BG, \Omega F_d] \xrightarrow{\cong} \langle (\mathbf{S}^1 \times BG) / (\{s_0\} \times BG), F_d \rangle \\ &\downarrow \\ \langle \mathbf{S}^1 \times BG, F_d \rangle &\xrightarrow{\cong} \langle B(\mathbb{Z} \times G), F_d \rangle \end{aligned} \quad (10)$$

and α is the unique map making the diagram commute. Here, s_0 is the basepoint of \mathbf{S}^1 . In diagram (9), each row is a naturally split short exact sequence, with splitting induced by the epimorphism $\mathbb{Z} \times G \twoheadrightarrow G$, $(i, g) \mapsto g$.

PROOF. See App. E. □

Corollary 5.4. Let G be any group. There are natural isomorphisms,

$$\tilde{h}^d(B(\mathbb{Z} \times G)) \cong h^{d-1}(BG) \oplus \tilde{h}^d(BG), \quad (11)$$

$$h^d(B(\mathbb{Z} \times G)) \cong h^{d-1}(BG) \oplus h^d(BG). \quad (12)$$

□

When h is the ordinary cohomology $H^\bullet(-; R)$ with coefficient ring R , Eq. (12) reduces to the familiar Künneth formula $H^\bullet(\mathbf{S}^1 \times BG; R) \cong H^\bullet(\mathbf{S}^1; R) \otimes_R H^\bullet(BG; R)$, where we recall that $H^i(\mathbf{S}^1; R) = R$ if $i = 0, 1$ and 0 otherwise. Eq. (12) is also easy to understand when h is a product of ordinary cohomology theories, i.e. $h^\bullet(-) = \prod_i H^{\bullet+d_i}(-; R_i)$ for arbitrary shifts d_i and rings R_i . In general, h can be a “twisted” product of ordinary theories, and may not have a cup product, so it is not obvious why Eq. (12) should hold. A handwavy argument would be to replace every R in the usual Künneth formula by $h^\bullet(\text{pt})$ and note that $h^\bullet(B\mathbb{Z}) \cong h^\bullet(\mathbf{S}^1) \cong h^\bullet(\text{pt}) \oplus h^{\bullet-1}(\text{pt})$ as graded discrete abelian groups.

5.3. A generalization to semidirect product $\mathbb{Z} \rtimes G$

In this subsection we generalize Proposition 5.3 to arbitrary semidirect products $\mathbb{Z} \rtimes G$. Recall, given any semidirect product $\mathbb{Z} \rtimes G$, that the composition of the canonical monomorphism $G \hookrightarrow \mathbb{Z} \rtimes G$ and the canonical epimorphism $\mathbb{Z} \rtimes G \twoheadrightarrow G$ is the identity on G . It follows that the induced map $BG \rightarrow B(\mathbb{Z} \rtimes G)$ is an embedding.

Proposition 5.5. Let G be any group and $\mathbb{Z} \rtimes G$ be any semidirect product. There is a natural commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{h}^d(B(\mathbb{Z} \rtimes G)/BG) & \xrightarrow{\tilde{\alpha}} & \tilde{h}^d(B(\mathbb{Z} \rtimes G)) & \xrightarrow{\tilde{\beta}} & \tilde{h}^d(BG) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{h}^d(B(\mathbb{Z} \rtimes G)/BG) & \xrightarrow{\alpha} & h^d(B(\mathbb{Z} \rtimes G)) & \xrightarrow{\beta} & h^d(BG) \longrightarrow 0 \end{array} \quad (13)$$

where $\tilde{\beta}$ and β are induced by the monomorphism $G \hookrightarrow \mathbb{Z} \rtimes G$, $g \mapsto (0, g)$, the two vertical maps are obtained by forgetting basepoints, $\tilde{\alpha}$ is induced by the quotient map $B(\mathbb{Z} \rtimes G) \rightarrow B(\mathbb{Z} \rtimes G)/BG$, and α is the unique map making the diagram commute. Here, BG denotes its homeomorphic image in $B(\mathbb{Z} \rtimes G)$ under the induced map $BG \rightarrow B(\mathbb{Z} \rtimes G)$. In diagram (13), each row is a naturally split short exact sequence, with splitting induced by the epimorphism $\mathbb{Z} \rtimes G \twoheadrightarrow G$, $(i, g) \mapsto g$.

PROOF. See App. E. □

Corollary 5.6. Let G be any group and $\mathbb{Z} \rtimes G$ be any semidirect product. There are natural isomorphisms,

$$\tilde{h}^d(B(\mathbb{Z} \rtimes G)) \cong \tilde{h}^d(B(\mathbb{Z} \rtimes G)/BG) \oplus \tilde{h}^d(BG), \quad (14)$$

$$h^d(B(\mathbb{Z} \rtimes G)) \cong \tilde{h}^d(B(\mathbb{Z} \rtimes G)/BG) \oplus h^d(BG). \quad (15)$$

□

5.4. A generalization to arbitrary product $G_1 \times G_2$

In this subsection we generalize Proposition 5.3 to arbitrary products $G_1 \times G_2$.

Proposition 5.7. Let G_1 and G_2 be any groups. There is a natural commutative diagram,

$$\begin{array}{ccc} \tilde{h}^d(B(G_1 \times G_2)) & \xrightarrow{\cong} & \tilde{h}^d(BG_1) \oplus \tilde{h}^d(BG_1 \wedge BG_2) \oplus \tilde{h}^d(BG_2) \\ \downarrow & & \downarrow \\ h^d(B(G_1 \times G_2)) & \xrightarrow{\cong} & \tilde{h}^d(BG_1) \oplus \tilde{h}^d(BG_1 \wedge BG_2) \oplus h^d(BG_2) \end{array} \quad (16)$$

with the vertical maps obtained by forgetting basepoints, such that the canonical inclusions

$$\tilde{h}^d(BG_1) \hookrightarrow \tilde{h}^d(B(G_1 \times G_2)), \quad (17)$$

$$\tilde{h}^d(BG_2) \hookrightarrow \tilde{h}^d(B(G_1 \times G_2)), \quad (18)$$

$$h^d(BG_2) \hookrightarrow h^d(B(G_1 \times G_2)) \quad (19)$$

are induced by the canonical epimorphisms $G_1 \times G_2 \twoheadrightarrow G_1$, $G_1 \times G_2 \twoheadrightarrow G_2$, and $G_1 \times G_2 \twoheadrightarrow G_2$, respectively, and that the canonical projections

$$\tilde{h}^d(B(G_1 \times G_2)) \twoheadrightarrow \tilde{h}^d(BG_1), \quad (20)$$

$$\tilde{h}^d(B(G_1 \times G_2)) \twoheadrightarrow \tilde{h}^d(BG_2), \quad (21)$$

$$h^d(B(G_1 \times G_2)) \twoheadrightarrow h^d(BG_2) \quad (22)$$

are induced by the canonical monomorphisms $G_1 \hookrightarrow G_1 \times G_2$, $G_2 \hookrightarrow G_1 \times G_2$, and $G_2 \hookrightarrow G_1 \times G_2$, respectively.

PROOF. See App. E. □

Corollary 5.8. Let G_1 and G_2 be any groups and 0 be the trivial group. There are natural isomorphisms,

$$\tilde{h}^d(B(G_1 \times G_2)) \cong \tilde{h}^d(BG_1) \oplus \tilde{h}^d(BG_1 \wedge BG_2) \oplus \tilde{h}^d(BG_2), \quad (23)$$

and

$$h^d(B(G_1 \times G_2)) \cong \tilde{h}^d(BG_1) \oplus \tilde{h}^d(BG_1 \wedge BG_2) \oplus h^d(BG_2) \quad (24)$$

$$\cong h^d(BG_1) \oplus \tilde{h}^d(BG_1 \wedge BG_2) \oplus \tilde{h}^d(BG_2) \quad (25)$$

$$\cong \tilde{h}^d(BG_1) \oplus \tilde{h}^d(BG_1 \wedge BG_2) \oplus \tilde{h}^d(BG_2) \oplus h^d(B0). \quad (26)$$

□

5.5. A generalization to arbitrary semidirect product $G_1 \rtimes G_2$

In this subsection we generalize Propositions 5.3, 5.5, and 5.7 to arbitrary semidirect products $G_1 \rtimes G_2$. Recall, given any semidirect product $G_1 \rtimes G_2$, that the composition of the canonical monomorphism $G_2 \hookrightarrow G_1 \rtimes G_2$ and the canonical epimorphism $G_1 \rtimes G_2 \twoheadrightarrow G_2$ is the identity on G_2 . It follows that the induced map $BG_1 \rightarrow B(G_1 \rtimes G_2)$ is an embedding.

Proposition 5.9. Let $G_1 \rtimes G_2$ be any semidirect product of any groups G_1 and G_2 . There is a natural commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{h}^d(B(G_1 \rtimes G_2)/BG_2) & \xrightarrow{\tilde{\alpha}} & \tilde{h}^d(B(G_1 \rtimes G_2)) & \xrightarrow{\tilde{\beta}} & \tilde{h}^d(BG_2) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{h}^d(B(G_1 \rtimes G_2)/BG_2) & \xrightarrow{\alpha} & h^d(B(G_1 \rtimes G_2)) & \xrightarrow{\beta} & h^d(BG_2) \longrightarrow 0 \end{array} \quad (27)$$

where $\tilde{\beta}$ and β are induced by the canonical monomorphism $G_2 \hookrightarrow G_1 \rtimes G_2$, the two vertical maps are obtained by forgetting basepoints, $\tilde{\alpha}$ is induced by the quotient map $B(G_1 \rtimes G_2) \rightarrow B(G_1 \rtimes G_2)/BG_2$, and α is the unique map making the diagram commute. Here, BG_2 denotes its homeomorphic image in $B(G_1 \rtimes G_2)$ under the induced map $BG_2 \rightarrow B(G_1 \rtimes G_2)$. In diagram (27), each row is a naturally split short exact sequence, with splitting induced by the canonical epimorphism $G_1 \rtimes G_2 \twoheadrightarrow G_2$.

PROOF. See App. E. □

Corollary 5.10. Let $G_1 \rtimes G_2$ be any semidirect product of any groups G_1 and G_2 . There are natural isomorphisms,

$$\tilde{h}^d(B(G_1 \rtimes G_2)) \cong \tilde{h}^d(B(G_1 \rtimes G_2)/BG_2) \oplus \tilde{h}^d(BG_2), \quad (28)$$

$$h^d(B(G_1 \rtimes G_2)) \cong \tilde{h}^d(B(G_1 \rtimes G_2)/BG_2) \oplus h^d(BG_2). \quad (29)$$

□

6. Consequences of the Hypothesis: Physical Implications

In this section we discuss physical implications of the mathematical results in Sec. 5. We stress that the results below are not specific to any classification proposal or physical dimension, and apply to the fermionic case as well as the bosonic case. Some of our results serve as comprehensive generalizations of special cases (which are typically proposal-, dimension-, or particle content-specific) that already exist in the literature, while others are entirely new. Occasionally, in order to paint a full physical picture, it is necessary to bring in assumptions in addition to the Hypothesis or take leaps of faith, but such assumptions or leaps will be kept to a minimum and always stated explicitly.

In what follows, we denote by h the generalized cohomology theory appearing in the Hypothesis, by \tilde{h} the corresponding reduced theory, and by $(F_d)_{d \in \mathbb{Z}}$ their defining Ω -spectrum.

6.1. Unification of old and new definitions of SPT phases

In Sec. 2.3, we reviewed the old definition of SPT phases [8, 31], and formalized a new definition of SPT phases based on ideas in Refs. [35, 38, 39, 41, 42, 55]. The old definition is in terms of deformability to product states, whereas the new one is in terms of invertibility of phases, which is closely related and potentially equivalent [35, 38, 39, 41, 42, 55] to the condition of unique ground state on arbitrary spatial slice and, in two dimensions, the condition of no nontrivial anyonic excitations.

We have seen in Sec. 2.3.3 that d -dimensional G -protected SPT phases in the old sense form a subset of those in the new sense. Here we would like to make their relationship more explicit.

Physical Result 1. If SPT phases (in the new sense) are classified by a generalized cohomology theory h as in the Hypothesis, then d -dimensional invertible topological orders (i.e. d -dimensional SPT phases protected by the trivial symmetry group) are classified by $h^d(\text{pt})$.

PROOF. This is a simple application of the Hypothesis: set G to be the trivial group 0 and recall that the classifying space of the trivial group, $B0$, is homotopy equivalent to the one-point set, pt . \square

The merit of Physical Result 1 lies in the fact that the value on a point, $h^d(\text{pt})$, is basic to any generalized cohomology theory h . Given an h , $h^d(\text{pt})$ is usually the simplest to compute. Conversely, from $h^d(\text{pt})$, one can deduce important information about $h^d(X)$ for any X (which was the basis of the approach in Refs. [40, 42, 43]; see Apps. A.3 and A.7).

Physical Result 2. If SPT phases in the new sense are classified by an unreduced generalized cohomology theory h as in the Hypothesis, then SPT phases in the old sense are classified by the corresponding reduced theory \tilde{h} , where the same remarks about additivity and functoriality apply.

PROOF. As remarked in Sec. 2.3.3, SPT phases in the old sense are precisely those SPT phases in the new sense that, by forgetting the symmetry, represent the trivial topological order. Thus, by the functoriality part of the Hypothesis, they are precisely the kernel of the map p in Lemma 5.1, which is naturally isomorphic to $\tilde{h}^d(BG)$ by exactness. \square

We would like to point out that the converse of Physical Result 2 is not automatic. That is, had we formulated the Hypothesis for SPT phases in the old sense in terms of \tilde{h} , then it would not have been nearly as easy, if not impossible, to deduce that SPT phases in the new sense are classified by h .

Physical Result 3. There is a natural isomorphism of abelian groups,

$$\left\{ \begin{array}{c} d\text{-dimensional} \\ \text{protected SPT phases} \\ \text{in the new sense} \end{array} \right\}^{G-} \cong \left\{ \begin{array}{c} d\text{-dimensional} \\ \text{protected SPT phases} \\ \text{in the old sense} \end{array} \right\}^{G-} \oplus \left\{ \begin{array}{c} d\text{-dimensional} \\ \text{invertible topo-} \\ \text{logical orders} \end{array} \right\}. \quad (30)$$

PROOF. We have seen in Physical Result 1 that $h^d(\text{pt}) \cong h^d(B0)$ classifies d -dimensional invertible topological orders, and in Physical Result 2 that $\tilde{h}^d(BG)$ classifies d -dimensional G -protected SPT phases in the old sense. The desired natural isomorphism then follows from Corollary 5.2. \square

We note that the special case of Physical Result 3 where h is the spin cobordism theory in Ref. [36] has been pointed out by Ref. [36].

The next result gives more information about the isomorphism in Physical Result 3.

Physical Result 4. The isomorphism in Physical Result 3 is such that the canonical injection

$$i : \left\{ \begin{array}{l} d\text{-dimensional } G\text{-protected} \\ \text{SPT phases in the old sense} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} d\text{-dimensional } G\text{-protected} \\ \text{SPT phases in the new sense} \end{array} \right\} \quad (31)$$

is given by inclusion, and that the canonical projection

$$p : \left\{ \begin{array}{l} d\text{-dimensional } G\text{-protected} \\ \text{SPT phases in the new sense} \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{l} d\text{-dimensional invertible} \\ \text{topological orders} \end{array} \right\} \quad (32)$$

is given by forgetting symmetry G .

PROOF. Recall that Corollary 5.2 came from Lemma 5.1. We have seen in Physical Result 1 that $h^d(B0)$ classifies d -dimensional invertible topological orders, and in Physical Result 2 that $\ker p$ classifies d -dimensional G -protected SPT phases in the old sense. The first half of Physical Result 4 is then trivial, whereas the second half follows from the functoriality part of the Hypothesis. \square

6.2. Strong and weak topological indices in the interacting world

As observed already in the 1-dimensional bosonic case [30, 31], the classification of SPT phases can be modified by an additionally imposed discrete spatial translational symmetry. Two translationally invariant systems that are inequivalent in the presence of translational symmetry may be deformable to each other via non-translationally invariant paths. A priori, it is also not obvious that there are no intrinsically non-translationally invariant SPT phases.

Here we would like to clarify the relationship between classifications in the presence and absence of discrete translational symmetry. We will begin with discrete translation \mathbb{Z} in only one direction and take G to be a symmetry it commutes with (hence forming $\mathbb{Z} \times G$). We shall assume that the Hypothesis is valid in this setup (see Sec. 2.1).

Physical Result 5. Let \mathbb{Z} act as discrete spatial translations. Then there is a natural isomorphism of abelian groups,

$$\left\{ \begin{array}{l} d\text{-dimensional} \\ (\mathbb{Z} \times G)\text{-protected} \\ \text{SPT phases} \end{array} \right\} \cong \left\{ \begin{array}{l} (d-1)\text{-dimensional} \\ G\text{-protected SPT} \\ \text{phases} \end{array} \right\} \oplus \left\{ \begin{array}{l} d\text{-dimensional} \\ G\text{-protected} \\ \text{SPT phases} \end{array} \right\}. \quad (33)$$

PROOF. This is an immediate consequence of the second isomorphism in Corollary 5.4. \square

The next two results give more information about the isomorphism.

Physical Result 6. The isomorphism in Physical Result 5 is such that the canonical projection

$$\beta : \left\{ \begin{array}{l} d\text{-dimensional } (\mathbb{Z} \times G)\text{-} \\ \text{protected SPT phases} \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{l} d\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \end{array} \right\}. \quad (34)$$

is given by forgetting translational symmetry.

PROOF. Recall that Corollary 5.4 came from Proposition 5.3. The claim then follows from the functoriality part of the Hypothesis. \square

Physical Result 7. It seems plausible that the isomorphism in Physical Result 5 is such that the canonical injection

$$\alpha : \left\{ \begin{array}{l} (d-1)\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} d\text{-dimensional } (\mathbb{Z} \times G)\text{-} \\ \text{protected SPT phases} \end{array} \right\} \quad (35)$$

is given by the layering construction where one produces a d -dimensional $(\mathbb{Z} \times G)$ -symmetric system by stacking identical copies of a $(d-1)$ -dimensional G -symmetric system.

ARGUMENTS. A special case of Physical Result 5 has been observed in the group cohomology classification of 1-dimensional bosonic SPT phases, where α is indeed given by such a layering construction; see Sec. VB4 of Ref. [30] and Sec. IVC3 of Ref. [31]. As for arbitrary generalized cohomology theories in arbitrary dimensions, a field-theoretic construction is proposed in App. B to justify this interpretation of α . \square

Therefore, in parallel with the notions of strong and weak topological insulators [83], we can divide d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phases into strong ones and weak ones, according to whether they can be produced through the layering construction, or equivalently whether they become trivial upon forgetting the translational symmetry. We shall call the first and second direct summands in the right-hand side of Eq. (33) the *weak topological index* and the *strong topological index*, respectively. Their counterparts in Ref. [83] would be $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and \mathbb{Z}_2 , respectively. Despite the similarities, there is a crucial distinction between our Physical Results 5-7 and Ref. [83]: the former deal with possibly interacting bosonic or fermionic systems, whereas the latter dealt with free fermion systems.

The next two addenda tell us how Physical Result 5 interacts with Physical Result 3.

Addendum to Physical Result 6. β does not mix different invertible topological orders. In particular, it takes SPT phases in the old sense to SPT phases in the old sense.

PROOF. The invertible topological order an SPT phase represents is obtained by forgetting all symmetry operations. We have seen that β is given by forgetting \mathbb{Z} . Since forgetting first \mathbb{Z} and then G is equivalent to forgetting $\mathbb{Z} \times G$ in one step, β must preserve invertible topological orders. The second half of the addendum also follows independently from the commutativity of the second square in Eq. (9). \square

Addendum to Physical Result 7. α can never produce d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phases with nontrivial invertible topological orders.

PROOF. This follows from the commutativity of the first square in Eq. (9). \square

This addendum is independent of the arguments for Physical Result 7. If one believes in those arguments, however, then what the addendum is saying is that the layering construction can never produce nontrivial invertible topological orders.

Now, let us spell out the implications of Physical Results 5-7 in detail.

Physical Result 8. Let \mathbb{Z} act as discrete spatial translations and assume the interpretation of α in Physical Result 7 is valid. Then we have the following:

- (i) Every d -dimensional G -protected SPT phase can be canonically represented by a d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phase.

- (ii) The layering construction turns equivalent $(d - 1)$ -dimensional systems into equivalent d -dimensional systems, and is hence well-defined at the level of phases.
- (iii) The layering construction commutes with addition of phases and replacement of G .
- (iv) The layering construction turns trivial, nontrivial, or distinct $(d - 1)$ -dimensional G -protected SPT phases into trivial, nontrivial, distinct d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phases, respectively.
- (v) Every d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phase obtained through the layering construction becomes trivial upon forgetting \mathbb{Z} .
- (vi) Every d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phase that becomes trivial upon forgetting \mathbb{Z} can be obtained through the layering construction.
- (vii) If two d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phases become the same phase upon forgetting \mathbb{Z} , then their difference can be obtained through the layering construction.
- (viii) A $(\mathbb{Z} \times G)$ -protected SPT phase is uniquely determined by its strong and weak topological indices, and every combination of strong and weak topological indices is allowed.

PROOF. All statements follow from the exactness of the second row of Eq. (9), except for the one about replacement of G , which depends on naturality, and the one about canonical representative, which depends on splitting. \square

The results here have been observed in the group cohomology classification of 1-dimensional bosonic SPT phases; see Sec. VB4 of Ref. [30] and Sec. IVC3 of Ref. [31]. Note that 0-dimensional G -protected SPT phases are nothing but isomorphism classes of 1-dimensional unitary representations of G , which are classified by $H_{\text{group}}^1(G; U(1)) \cong H^2(BG; \mathbb{Z})$ (only finite groups were considered in Refs. [30, 31]).

6.3. Hierarchy of strong and weak topological indices

We now perform a sanity check on Physical Results 5-8 by imposing discrete spatial translational symmetry in multiple linearly independent directions. With translational symmetry in two directions, for example, we have

$$\begin{aligned}
\left\{ \begin{array}{l} d\text{-dim } (\mathbb{Z} \times \mathbb{Z} \times \\ G\text{-SPT phases} \end{array} \right\} &\cong \{ (d-1)\text{-dim } (\mathbb{Z} \times G)\text{-SPT phases} \} \oplus \{ d\text{-dim } (\mathbb{Z} \times G)\text{-SPT phases} \} \\
&\cong \left\{ \begin{array}{l} (d-2)\text{-dim} \\ G\text{-SPT} \\ \text{phases} \end{array} \right\} \oplus \left\{ \begin{array}{l} (d-1)\text{-dim} \\ G\text{-SPT} \\ \text{phases} \end{array} \right\} \oplus \left\{ \begin{array}{l} (d-1)\text{-dim} \\ G\text{-SPT} \\ \text{phases} \end{array} \right\} \oplus \left\{ \begin{array}{l} d\text{-dim} \\ G\text{-SPT} \\ \text{phases} \end{array} \right\}, \quad (36)
\end{aligned}$$

where we have abbreviated “ d -dimensional G -protected SPT phases” to “ d -dim G -SPT phases.” In the last line, the last direct summand is a strong index arising from forgetting translational symmetry in both directions; the second and third direct summands are weak indices corresponding to stacking identical copies of 1-codimensional phases in two different ways; the first direct summand is a “very weak” index corresponding to stacking 2-codimensional phases two-dimensionally. This decomposition can be generalized to translation in n directions in a straightforward fashion.

Physical Result 9. Let \mathbb{Z}^n act as discrete spatial translations in n linearly independent directions. Then there is a natural isomorphism of abelian groups,

$$\begin{aligned}
\{d\text{-dim } (\mathbb{Z}^n \times G)\text{-SPT phases}\} &\cong \{(d-n)\text{-dim } G\text{-SPT phases}\} \\
&\oplus \underbrace{\left\{ \begin{array}{c} (d-n+1)\text{-dim} \\ G\text{-SPT phases} \end{array} \right\} \oplus \cdots \oplus \left\{ \begin{array}{c} (d-n+1)\text{-dim} \\ G\text{-SPT phases} \end{array} \right\}}_{\binom{n}{n-1} = n \text{ times}} \\
&\dots \\
&\oplus \underbrace{\left\{ \begin{array}{c} (d-k)\text{-dim } G\text{-} \\ \text{SPT phases} \end{array} \right\} \oplus \cdots \oplus \left\{ \begin{array}{c} (d-k)\text{-dim } G\text{-} \\ \text{SPT phases} \end{array} \right\}}_{\binom{n}{k} \text{ times}} \\
&\dots \\
&\oplus \underbrace{\left\{ \begin{array}{c} (d-1)\text{-dim } G\text{-} \\ \text{SPT phases} \end{array} \right\} \oplus \cdots \oplus \left\{ \begin{array}{c} (d-1)\text{-dim } G\text{-} \\ \text{SPT phases} \end{array} \right\}}_{\binom{n}{1} = n \text{ times}} \\
&\oplus \{d\text{-dim } G\text{-SPT phases}\}, \tag{37}
\end{aligned}$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$.

PROOF. Iterate Physical Result 5. □

We thus see a hierarchy of topological indices in different codimensions. There is a single strong topological index, in 0 codimension (i.e. d dimensions), which arises from forgetting translational symmetry in all n directions. There are $\binom{n}{k}$ weak topological indices in k codimensions (i.e. $d - k$ dimensions), which correspond to stacking identical copies of k -codimensional phases in $\binom{n}{k}$ different ways. This hierarchy is visualized in Fig. 6.

6.4. Pumping, Floquet eigenstates, and classification of Floquet SPT phases

Here we would like to reinterpret the \mathbb{Z} in Physical Results 5-8 as a discrete temporal translational symmetry. Accordingly, we shall call a $(\mathbb{Z} \times G)$ -protected SPT phase a G -protected Floquet SPT phase. As usual, we allow for interactions.

A few words about the definition of Floquet SPT phases are in order. In essence, what we would like to define as a G -protected Floquet SPT phase is a deformation class of Floquet eigenstates, rather than a deformation class of periodic Hamiltonians [16]. A Floquet eigenstate is invariant under both the Floquet operator $\exp\left[-i \int_0^T \hat{H}(t) dt\right]$ and the G -action, which makes it clear what it would mean to forget the discrete temporal translational symmetry. In principle, different Floquet eigenstates of the same periodic Hamiltonian can represent different G -protected Floquet SPT phases, and different periodic Hamiltonians can have common Floquet eigenstates. We shall assume that the Hypothesis is valid for discrete temporal translational symmetry with respect to this notion of Floquet SPT phases (see Sec. 2.1).

The results below mirror the results in Sec. 6.2.

Physical Result 10. Let G act in a way that commutes with the group \mathbb{Z} of discrete temporal translations. There is a natural isomorphism of abelian groups,

$$\left\{ \begin{array}{c} d\text{-dimensional } G\text{-protected} \\ \text{Floquet SPT phases} \end{array} \right\} \cong \left\{ \begin{array}{c} (d-1)\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \end{array} \right\} \oplus \left\{ \begin{array}{c} d\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \end{array} \right\}. \tag{38}$$

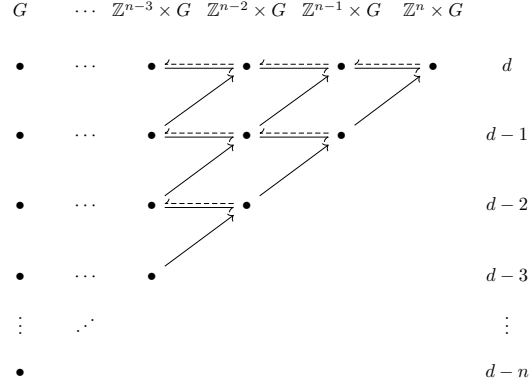


Figure 6: Illustration of the hierarchy of strong and weak topological indices. Different rows and columns correspond to different dimensions and symmetry groups, respectively. \mathbb{Z}^n acts as discrete spatial translations in n linearly independent directions, where $n \leq d$. Each dot denotes an abelian group of SPT phases of the appropriate dimension protected by the appropriate symmetry group. An upper-rightward move corresponds to the layering construction in the relevant direction (α in Physical Result 7). A horizontal leftward move corresponds to forgetting translational symmetry in the relevant direction (β in Physical Result 6). A horizontal rightward move corresponds to taking the canonical SPT phase with one additional translational symmetry [the splitting of the second row in Eq. (9)]. Each path along solid arrows from the leftmost column to the rightmost dot contributes a topological index, with the horizontal path responsible for the strong topological index and the rest responsible for the weak topological indices.

PROOF. Same as Physical Result 5. □

Physical Result 11. The isomorphism in Physical Result 10 is such that the canonical projection

$$\beta : \left\{ \begin{array}{l} d\text{-dimensional } G\text{-protected} \\ \text{Floquet SPT phases} \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{l} d\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \end{array} \right\}. \quad (39)$$

is given by forgetting temporal translational symmetry.

PROOF. Same as Physical Result 6. □

Physical Result 12. It seems plausible that the isomorphism in Physical Result 10 is such that the canonical projection

$$\gamma : \left\{ \begin{array}{l} d\text{-dimensional } G\text{-protected} \\ \text{Floquet SPT phases} \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{l} (d-1)\text{-dimensional } G\text{-} \\ \text{protected SPT phases} \end{array} \right\} \quad (40)$$

is given by measuring what $(d-1)$ -dimensional G -protected SPT phase is pumped across an imaginary cut in a d -dimensional system in one Floquet cycle.

ARGUMENTS. A special case of Physical Result 10 has been observed in the classification of 1-dimensional Floquet SPT phases within the group cohomology framework [15–17], where it was argued that γ should have such a pumping interpretation, at least when G is finite abelian. □

We note that special cases of Physical Results 10–12 where h is the group cohomology theory in Ref. [33] have appeared in classifications of 1-dimensional bosonic Floquet SPT phases [15–17], in which

the fermionic case was also discussed.

The next two addenda tell us how Physical Result 10 interacts with Physical Result 3.

Addendum to Physical Result 11. β does not mix different invertible topological orders. In particular, it takes Floquet SPT phases in the old sense to SPT phases in the old sense.

PROOF. Same as Addendum to Physical Result 6. □

Addendum to Physical Result 12. Every $(d-1)$ -dimensional G -protected SPT phase, with trivial or nontrivial invertible topological order, can be produced via γ from a d -dimensional G -protected Floquet SPT phase with trivial invertible topological order.

PROOF. Same as Addendum to Physical Result 7. This addendum is independent of the arguments for Physical Result 12. □

Now, let us spell out the implications of Physical Results 10-12 in detail.

Physical Result 13. Let G act in a way that commutes with the group \mathbb{Z} of discrete temporal translations and assume the interpretation of γ in Physical Result 12 is valid. Then we have the following:

- (i) Equivalent Floquet systems pump equivalent stationary systems across the cut. That is, pumping is well-defined at the level of phases.
- (ii) Every d -dimensional G -protected SPT phase can be obtained by forgetting the discrete temporal translational symmetry of some canonical d -dimensional G -protected Floquet SPT phase, which pumps the trivial $(d-1)$ -dimensional G -protected SPT phase across the cut.
- (iii) Every $(d-1)$ -dimensional G -protected SPT phase can be obtained through pumping from some canonical d -dimensional G -protected Floquet SPT phase, which becomes trivial upon forgetting the discrete temporal translational symmetry.
- (iv) Pumping commutes with addition of phases. That is, the $(d-1)$ -dimensional G -protected SPT phase pumped across the cut by the sum of two d -dimensional G -protected Floquet SPT phases is equal to the sum of the $(d-1)$ -dimensional G -protected SPT phases that are pumped across the cut by the two d -dimensional G -protected Floquet SPT phases respectively.
- (v) Pumping commutes with replacement of G . That is, given a homomorphism $\varphi : G' \rightarrow G$ and a d -dimensional G -protected Floquet SPT phase $[c]$, if we write $\varphi^*[c]$ for the d -dimensional G' -protected Floquet SPT phase induced from $[c]$ via φ , then the $(d-1)$ -dimensional G' -protected SPT phase pumped across the cut by $\varphi^*[c]$ is equal to the one induced via φ from the $(d-1)$ -dimensional G -protected SPT phase that is pumped across the cut by $[c]$.
- (vi) A d -dimensional G -protected Floquet SPT phase is uniquely determined by
 - (a) the d -dimensional G -protected SPT phase obtained by forgetting the discrete temporal translational symmetry and
 - (b) the $(d-1)$ -dimensional G -protected SPT phase pumped across the cut,
and every combination of d - and $(d-1)$ -dimensional G -protected SPT phases is allowed.

PROOF. All statements follow from the exactness and splitting of the second row of Eq. (9), except for the one about replacement of G , which depends on naturality. \square

One can imagine combining ideas in Secs. 6.2 and 6.4 to treat cases where both spatial and temporal translational symmetries are present, cases where only a combination of spatial and temporal translations is a symmetry, etc., i.e., in a loose sense, spacetime crystals [84–86].

6.5. Applications to space group-protected SPT phases

A growing body of evidence [30, 31, 57–65] has emerged suggesting that the Generalized Cohomology Hypothesis is applicable to most, perhaps all, non-on-site symmetries as well as on-site ones. Namely, if G is a symmetry group acting in a microscopically unitary, possibly non-on-site, but orientation-preserving fashion, then the classification of d -dimensional G -protected SPT phases is given by $h^d(BG)$ for the same generalized cohomology theory h one would use to classify SPT phases protected by on-site unitary symmetries¹⁰. Systematic investigation of this principle has been put forth by Refs. [64, 65]. In Ref. [64], the principle was demonstrated for bosonic SPT phases in 1, 2, and 3 dimensions through a tensor network construction. In Ref. [65], it was demonstrated through a space group “gauging” procedure proposed therein. The principle is in accord with previously discovered special cases [30, 31, 57–63]. While some discrepancies exist [59, 60, 62, 74], we suspect they are due to the inhomogeneous definitions of space group-protected SPT phases in the literature; they also involve effectively antiunitary symmetry operations, which are beyond the purview of this paper.

With these remarks out of the way, let us see what implications the results in Sec. 5 have on SPT phases protected by space group symmetries.

Physical Result 14. Let SG be a space group with all orientation-reversing elements removed. Let PG be its point group. If SG is symmorphic, then every d -dimensional PG -protected SPT phase can be canonically represented by a d -dimensional SG -protected SPT phase.

PROOF. When symmorphic, $SG \cong TG \rtimes PG$, where TG is the translational group. Apply Proposition 5.9. \square

Put differently, when a space group is symmorphic, lifting the translational symmetry can never lead to “intrinsically new” phases. Note that one is not obligated to retain all orientation-preserving elements in the symmetry group of a physical lattice. It is perfectly fine to let G contain only rotations about a particular axis, for instance.

Physical Result 15. Let G_0 be a group that acts in an on-site fashion and SG be a space group with all orientation-reversing elements removed. Then every d -dimensional SG -protected SPT phase can be canonically represented by a d -dimensional $(G_0 \rtimes SG)$ -protected SPT phase.

PROOF. Apply Proposition 5.9. \square

Again, Physical Result 15 says, given an on-site symmetry and a space group symmetry, that lifting the former can never lead to “intrinsically new” phases. Note that there is no condition on symmorphism. An example of G_0 and SG that do not commute is this: suppose $G_0 = \mathbb{Z}_n$ is generated by spin rotation

¹⁰The claim is actually more general, in that one can allow for orientation-reversing (e.g. parity) or microscopically antiunitary (e.g. time-reversal) symmetry operations, as long as both are treated antiunitarily [64, 65]. In our framework, this would give rise to a nontrivial action on the Ω -spectrum, thereby necessitating twisted generalized cohomology theories. Since we are only concerned with effectively unitary symmetry actions, we have simplified the claim a little.

about the y -axis by an angle of $\frac{2\pi}{n}$, and $SG = \mathbb{Z}_2$ is generated by spatial rotation about the z -axis by an angle of π ; then the two does not commute as long as $n > 2$.

When G_0 happens to commute with SG , we have the following additional result.

Physical Result 16. With the same set-up as in Physical Result 15, if SG commutes with G_0 , then every d -dimensional G_0 -protected SPT phase can be canonically represented by a d -dimensional $(G_0 \times SG)$ -protected SPT phase.

PROOF. Apply Proposition 5.7 or 5.9. □

On the other hand, if SG happens to be symmorphic, we have the following result.

Physical Result 17. With the same set-up as in Physical Result 15, if SG is symmorphic, then every d -dimensional PG -protected SPT phase can be canonically represented by a d -dimensional $(G_0 \rtimes PG)$ -protected SPT phase, in fact a d -dimensional $(G_0 \rtimes SG)$ -protected one, where PG is the point group.

PROOF. When SG is symmorphic, the total symmetry group is $G_0 \rtimes SG \cong (G_0 \times TG) \rtimes PG$, where TG is the translational group. Apply Proposition 5.9. □

This says, given an on-site symmetry and a symmorphic space group symmetry, that lifting the on-site symmetry and the translational symmetry can never lead to “intrinsically new” phases.

Finally, let us see how Physical Results 14-17 interact with Physical Result 3.

Addendum to Physical Results 14-17. If the phase being represented has trivial invertible topological order, then so does the canonical phase that represents it.

PROOF. This follows from the commutativity of the second square in Eq. (13). □

6.6. Obstruction-free enlargement of symmetry group

Here we would like to discuss the enlargement of symmetry groups in general. Let $G' \subset G$ be a subgroup. As one replaces G' by G , one expects to refine the classification of SPT phases. It is also possible, however, for certain G' -protected SPT phases to be eliminated, for a priori there may be obstructions to lifting an action of G' over to G . Here we give a sufficient condition for the absence of such obstructions.

Physical Result 18. Given $G' \subset G$, if there exists a subgroup $G'' \subset G$ such that G is a semidirect product $G'' \rtimes G'$, then every d -dimensional G' -protected SPT phase can be representable by a d -dimensional G -protected SPT phase.

PROOF. The condition is equivalent to the existence of a homomorphism $\pi : G \rightarrow G'$ such that $\pi \circ \iota = \text{id}$, where $\iota : G' \hookrightarrow G$ is the inclusion. This implies that $\iota^* \circ \pi^* : h^d(BG') \rightarrow h^d(BG) \rightarrow h^d(BG')$ is the identity. In particular, $\iota^* : h^d(BG) \rightarrow h^d(BG')$ is surjective. □

Note that direct products are considered to be special cases of semidirect products. Moreover, there are many equivalent criteria for when G is such a semidirect product:

- (i) There exists a normal subgroup $G'' \subset G$ such that every element $g \in G$ can be written as $g = g''g'$ for some unique $g'' \in G''$ and $g' \in G'$.
- (ii) There exists a normal subgroup $G'' \subset G$ such that every element $g \in G$ can be written as $g = g'g''$ for some unique $g' \in G'$ and $g'' \in G''$.
- (iii) There exists a surjective homomorphism $G \rightarrow G'$ that is the identity on G' .

As a special case, the enlargement from the trivial symmetry group to any symmetry group G is always obstruction-free. That is, every invertible topological order can be represented some G -protected SPT phase. This fact has been surreptitiously incorporated into Fig. 4.

7. Summary and Outlook

We have taken a novel, minimalist approach to the classification problem of SPT phases, where instead of directly classifying SPT phases, we looked for common ground among various existing classification proposals, which gave conflicting predictions in certain cases. The key in this approach was the formulation of a Generalized Cohomology Hypothesis that was satisfied by various proposals and captured essential aspects of SPT classification. We took the Hypothesis as the starting point and derived rigorous, general results from it. These results were born to be independent of which proposal is correct (or whether any proposal is correct at all, as long as the unknown complete classification satisfies the Hypothesis, which seems plausible on independent grounds). They typically give relations between classifications of SPT phases in different dimensions and/or protected by different symmetry groups. They hold in arbitrarily high dimensions and apply equally to fermionic and bosonic SPT phases. Our formalism works not only for on-site symmetries but also, as we argued, for discrete temporal translation, discrete spatial translation, and other space group symmetries. In a sense, what we have accomplished was not a classification, but rather a meta-, or second-order classification of SPT phases, and the merit of this approach lies in the unprecedented universality of our results.

We believe the results presented herein are only the tip of an iceberg. Generalized cohomology theories, and by extension infinite loop spaces and stable homotopy theory [75, 76], are well-studied mathematical subjects with plenty of theorems one can draw from. An effort to understand these subjects should prove worthwhile. As another step in the same direction, we will derive and interpret the following results in an upcoming paper:

- (i) Given coprime positive integers m and n , we have $\tilde{h}^d(B\mathbb{Z}_{mn}) \cong \tilde{h}^d(B(\mathbb{Z}_m \times \mathbb{Z}_n)) \cong \tilde{h}^n(B\mathbb{Z}_m) \oplus \tilde{h}^n(B\mathbb{Z}_n)$ regardless of \tilde{h} .
- (ii) There exist nontrivial discrete groups G for which $\tilde{h}^d(BG) = 0$ for all d regardless of \tilde{h} .
- (iii) There exist non-isomorphic finite groups G_1, G_2 for which $h^d(BG_1) \cong h^d(BG_2)$ regardless of h , at least in low dimensions with an additional, well-founded physical input.

Let us conclude with some interesting open questions.

- (i) How would our results generalize if effectively antiunitary symmetries were allowed, which would give rise to group actions on the Ω -spectrum and necessitate twisted generalized cohomology theories?
- (ii) Does the multiplicative structure of a multiplicative generalized cohomology theory have a physical meaning¹¹?
- (iii) Do generalized cohomology groups in negative dimensions have a physical meaning¹²?
- (iv) Can the Hypothesis be derived from “first principles”?

¹¹We thank Ammar Husain for suggesting this.

¹²We thank Ashvin Vishwanath for suggesting this.

- (v) What is the counterpart of generalized cohomology theories for topological orders, or more generally G -protected topological phases?

A. Existing Classification Proposals as Generalized Cohomology Theories

In this appendix, we explain how various proposals for the classification of SPT phases can be viewed as generalized cohomology theories. Below, we denote by $K(A, n)$ the n -th Eilenberg-Mac Lane space of A (see App. F.4).

A.1. Borel group cohomology proposal

Ref. [33] proposed that d -dimensional G -protected bosonic SPT phases are classified by $H_{\text{group}}^{d+1}(G; U(1))$ when G is finite and acts in an on-site, unitary fashion. Here, $H_{\text{group}}^{\bullet}$ denotes group cohomology. For infinite or continuous groups, Ref. [33] conjectured a classification by a Borel group cohomology group $H_{\text{Borel}}^{d+1}(G; U(1))$, which is naturally isomorphic to $H^{d+2}(BG; \mathbb{Z})$ [87]. Here, $H^{\bullet}(-; \mathbb{Z})$ is the ordinary (topological) cohomology theory with \mathbb{Z} coefficient [78]. Ordinary cohomology theories are the most ordinary kind of generalized cohomology theories. We know from Table 2 that they are represented by Eilenberg-Mac Lane spectra. Taking into account the shift in dimension, we thus have

$$H_{\text{Borel}}^{d+1}(G; U(1)) \cong H^{d+2}(BG; \mathbb{Z}) \cong [BG, K(\mathbb{Z}, d+2)]. \quad (41)$$

It can be seen either at the level of Ω -spectrum or by inspecting Definitions F.55 and F.56 that a shift in dimension turns generalized cohomology theories into generalized cohomology theories.

We will prove in App. D that the discrete abelian group and functorial structures of $H_{\text{group}}^{d+1}(G; U(1))$ for finite G correspond to stacking phases and replacing symmetry groups, respectively. This cannot be done for continuous or infinite discrete groups since no explicit construction was given in those cases.

It only remains to show that the $H^{d+2}(BG; \mathbb{Z})$ reduces to $H_{\text{group}}^{d+1}(G; U(1))$ in physical dimensions $d \geq 0$ when G is finite. By comparing the definitions of group cohomology and cellular cohomology, one finds a natural isomorphism $H_{\text{group}}^{d+1}(G; U(1)) \cong H^{d+1}(BG; U(1))$ for discrete groups and in particular finite groups. Since $H^{d+1}(-; A) = \tilde{H}^{d+1}(-; A)$ for all $d \geq 0$ and coefficients A , the following lemma completes the proof.

Lemma A.1. For each $n \in \mathbb{Z}$, there is a natural transformation,

$$\tilde{H}^n(X; U(1)) \rightarrow \tilde{H}^{n+1}(X; \mathbb{Z}), \quad (42)$$

that is an isomorphism when $X = BG$ and G is a finite¹³.

PROOF. See App. E. □

A.2. Oriented cobordism proposal

Ref. [35] proposed that d -dimensional G -protected bosonic SPT phases are classified by

$$\text{Hom}(MSO_{d+1}(BG), U(1)) \quad (43)$$

when G is finite and acts in an on-site, unitary fashion. Here, $MSO_{\bullet}(X)$ denotes the n -th oriented bordism group, which is a discrete abelian group, of topological space X . Continuous symmetry groups were not dealt with in Ref. [35]. In fact, the proposal was to further quotient out a subgroup of “continuous theta-parameters,” but we may as well do a classification with such parameters allowed and quotient them out at the end of the day. Ref. [35] also assumed a “vanishing thermal Hall response,” but that is a matter of what the word “system” means, which was put in a black box in Sec. 2.3.2.

To prove that the oriented cobordism proposal is a generalized cohomology theory, it is best to use the algebraic definitions F.55 and F.56 of generalized cohomology theories, and the analogous algebraic definitions [78] of generalized homology theories, rather than the topological definitions 3.2 and 3.1. By inspecting these algebraic definitions, one can convince themselves that the functor $\text{Hom}(-, U(1))$ turns

¹³This result was stated informally without proof in Ref. [71].

generalized homology theories into generalized cohomology theories. The only axiom that is perhaps nontrivial to check is the exactness axiom, for which one should invoke the fact that $U(1)$ is an injective \mathbb{Z} -module. Knowing that oriented bordism MSO_\bullet is a generalized homology theory [75, 76, 88], we conclude that the oriented cobordism proposal is a generalized cohomology theory.

It can only be partially verified that the additive and functorial structures of the oriented cobordism proposal correspond to stacking phases and replacing symmetry groups, respectively, as no lattice model was given in Ref. [35].

Eq. (43) is different from the standard oriented cobordism group $MSO^{d+1}(BG)$ [35], and hence is not represented by the Thom spectrum MSO in the sense of Theorem F.57. It is, however, still related to the Thom spectrum MSO as oriented bordism groups $MSO_{d+1}(BG)$ can be defined in terms of it [75, 76, 88].

A.3. Kitaev's bosonic proposal

Kitaev's proposal [40, 42] is unique among all existing classification proposals for bosonic SPT phases. He took the Generalized Cohomology Hypothesis as a fundamental assumption and tried to construct an Ω -spectrum from physical knowledge. The key observation there was that $h^d(\text{pt})$'s simultaneously classify invertible topological orders (see Physical Result 1) and determine homotopy groups of the Ω -spectrum:

$$h^d(\text{pt}) \cong \pi_i(F_{i+d}) =: \pi_{-d}(F), \quad \forall i \quad (44)$$

Homotopy groups carry important information about a topological space. The additional information needed to determine the homotopy type of a space is given by so-called k -invariants [78], and they are sometimes unique for trivial reasons. Given the homotopy groups and k -invariants of a space, the reconstruction proceeds by building a Postnikov tower from the bottom up [78].

Refs. [40, 42] assumed that

$$F_0 \approx \mathbb{C}P^\infty, \quad h^1(\text{pt}) = 0, \quad h^2(\text{pt}) \cong \mathbb{Z}, \quad h^3(\text{pt}) = 0, \quad (45)$$

where $\mathbb{C}P^\infty$ is the space of rays of (the direct limit of) finite-dimensional Hilbert spaces (recall Sec. 4.6), and $h^2(\text{pt})$ is generated by the E_8 -model [45, 70, 71]. Physically, $\pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ can be identified with the integral of the Berry curvature, and $h^2(\text{pt}) \cong \mathbb{Z}$ can be identified with chiral central charge [40, 42]. Accordingly, the homotopy groups of the Ω -spectrum are

i	< -3	-3	-2	-1	0	1	2	> 2
$\pi_i(F)$?	0	\mathbb{Z}	0	0	0	\mathbb{Z}	0

Having a single nontrivial homotopy group, the homotopy type of F_1 can be trivially determined:

$$F_1 = K(\mathbb{Z}, 3), \quad (46)$$

since there is no k -invariant to worry about. It turns out that the homotopy type of F_2 can also be determined:

$$F_2 = K(\mathbb{Z}, 4) \times \mathbb{Z}, \quad (47)$$

but for that one must utilize the fact that $F_2 \simeq \Omega F_3$ is a loop space. Though not mentioned in Refs. [40, 42], we can go on to determine the homotopy type of F_3 . It has two nontrivial homotopy groups in positive dimensions and one potentially nontrivial k -invariant, which takes value in $H^6(K(\mathbb{Z}, 1); \mathbb{Z})$. Incidentally, $H^6(K(\mathbb{Z}, 1); \mathbb{Z}) = 0$, so this k -invariant must be trivial as well, and the homotopy type of F_3 can be determined:

$$F_3 = K(\mathbb{Z}, 5) \times K(\mathbb{Z}, 1) \simeq K(\mathbb{Z}, 5) \times \mathbf{S}^1. \quad (48)$$

A similar argument ($H^7(K(\mathbb{Z}, 2); \mathbb{Z}) = 0$ plus the fact that it is a loop space) shows that

$$F_4 = K(\mathbb{Z}, 6) \times K(\mathbb{Z}, 2) \times \pi_{-4}(F) \simeq K(\mathbb{Z}, 6) \times \mathbb{C}P^\infty \times h^4(\text{pt}), \quad (49)$$

but $h^4(\text{pt})$ is unknown. All higher dimensional F_d 's require further input.

It can only be partially verified that the additive and functorial structures of Kitaev's bosonic proposal correspond to stacking phases and replacing symmetry groups, respectively, as the lattice model given in Ref. [71] was schematic.

A.4. Freed's bosonic proposal

We refer the reader to Refs. [38, 39] in view of the complexity of the proposal.

A.5. Group supercohomology proposal

Ref. [34] proposed, when G is finite and acts in an on-site, unitary fashion, that d -dimensional G -protected fermionic SPT phases are classified by a group supercohomology group whose cochains of are pairs¹⁴

$$\nu_d : G^{d+1} \rightarrow U(1), \quad (51)$$

$$n_{d-1} : G^d \rightarrow \mathbb{Z}_2 \subset U(1). \quad (52)$$

We believe that the proposal amounts to using the Ω -spectrum with the homotopy groups

$$\pi_{i-d}(F) := \pi_i(F_d) \cong \begin{cases} \mathbb{Z}_2, & i = d, \\ \mathbb{Z}, & i = d + 2, \\ 0, & \text{otherwise,} \end{cases} \quad (53)$$

and the k -invariants (see App. A.3) defined as follows. Having at most two nontrivial homotopy groups, each F_d has at most one nontrivial k -invariant, k_{d+1} . If we denote by β and β' the Bockstein homomorphisms [78] associated with the first and second rows of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{\text{mod } 2} & \mathbb{Z}_2 \longrightarrow 0 \\ & & \parallel & & \downarrow \times \frac{1}{2} & & \downarrow e^{i\pi x} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \xrightarrow{e^{i2\pi x}} & U(1) \longrightarrow 0 \end{array} \quad (54)$$

and by τ the map induced by the last vertical map, then k_{d+1} is defined to be the unique map making the following diagram commute:

$$\begin{array}{ccc} H^d(-; \mathbb{Z}_2) & \xrightarrow{k_{d+1}} & H^{d+3}(-; \mathbb{Z}) \\ Sq^2 \downarrow & \nearrow \beta & \uparrow \beta' \\ H^{d+2}(-; \mathbb{Z}_2) & \xrightarrow{\tau} & H^{d+2}(-; U(1)) \end{array} \quad (55)$$

where Sq^2 is the Steenrod square [78], which Ref. [33] mentioned in passing. In other words,

$$k_{d+1} = \beta \circ Sq^2. \quad (56)$$

One can think of the resulting theory as some sort of “twisted product” between $H^{d+2}(-; \mathbb{Z})$ and $H^d(-; \mathbb{Z}_2)$, which should correspond to ν_d and n_{d-1} , respectively (recall Lemma A.1). Indeed, if all k_{d+1} 's were trivial, then F_d would simply be a product $K(\mathbb{Z}, d+2) \times K(\mathbb{Z}_2, d)$ and the generalized cohomology group would simply be $H^{d+2}(-; \mathbb{Z}) \oplus H^d(-; \mathbb{Z}_2)$. In reality, this is true in $d = 0, 1$ but not necessarily higher dimensions. Thus, we have

$$F_0 = K(\mathbb{Z}, 2) \times \mathbb{Z}_2 \simeq \mathbb{C}P^\infty \times \mathbb{Z}_2, \quad (57)$$

$$F_1 = K(\mathbb{Z}, 3) \times K(\mathbb{Z}_2, 1) \simeq K(\mathbb{Z}, 3) \times \mathbb{R}P^\infty, \quad (58)$$

¹⁴The cochains in Ref. [34] are actually triples

$$(\nu_d, n_{d-1}, u_{d-1}) \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G^{d+1}], U(1)) \times \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G^d], H^1(\mathbb{Z}_2^f, U(1))) \times \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G^d], H_{\text{group}}^1(G_f, U(1))), \quad (50)$$

where G_f is the full symmetry group including fermion parity, but at the level of equivalence classes, u_{d-1} is irrelevant. See App. C of Ref. [34].

while F_d with $d \geq 2$ has to be obtained as a pull-back along k_{d+1} :

$$\begin{array}{ccc} F_d & \dashrightarrow & PK(\mathbb{Z}, d+3) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}_2, d) & \xrightarrow{k_{d+1}} & K(\mathbb{Z}, d+3) \end{array} \quad (59)$$

where the vertical arrow on the right is the path space fibration (see App. F.1).

A.6. Spin cobordism proposal

We refer the reader to Ref. [36] in view of the complexity of the proposal.

A.7. Kitaev's fermionic proposal

Kitaev's proposal [42, 43] for the classification of fermionic SPT phases was in close analogy with the bosonic case discussed in App. A.3. Again, he took the Generalized Cohomology Hypothesis as a fundamental assumption and tried to construct an Ω -spectrum from physical knowledge. This time, it was assumed that

$$F_0 \approx \mathbb{C}P^\infty \times \mathbb{Z}_2, \quad h^1(\text{pt}) \cong \mathbb{Z}_2, \quad h^2(\text{pt}) \cong \mathbb{Z}, \quad (60)$$

where $\mathbb{C}P^\infty$ is the space of rays of (the direct limit of) finite-dimensional Hilbert spaces (recall Sec. 4.6), the \mathbb{Z}_2 in F_0 is fermion parity, the \mathbb{Z}_2 in $h^1(\text{pt})$ is generated by the Majorana chain [66], and \mathbb{Z} is generated by $(p+ip)$ -superconductors [67–69]. Physically, $\pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ can be identified with the integral of the Berry curvature. Accordingly, the homotopy groups of the Ω -spectrum are

i	< -2	-2	-1	0	1	2	> 2
$\pi_i(F)$?	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}	0

Unfortunately, without further input, one can only determine the homotopy type of F_d for $d \leq 0$. As for F_1 , there are two path components, which are homotopy equivalent since F_1 is a loop space. The component containing the basepoint has two nontrivial homotopy groups and one potentially nontrivial k -invariant,

$$k_2 \in H^4(K(\mathbb{Z}_2, 1); \mathbb{Z}) \cong \mathbb{Z}_2. \quad (61)$$

Thus there are two possibilities:

$$F_1 = X_3 \times \mathbb{Z}_2, \quad (62)$$

where X_3 is either $K(\mathbb{Z}, 3) \times K(\mathbb{Z}_2, 1)$ corresponding to $k_2 = 0$, or a more complicated space corresponding to $k_2 \neq 0$. If one borrows k_2 from App. A.5, then $k_2 = 0$, and $F_1 = K(\mathbb{Z}, 3) \times K(\mathbb{Z}_2, 1) \times \mathbb{Z}_2 \simeq K(\mathbb{Z}, 3) \times \mathbb{C}P^\infty \times \mathbb{Z}_2$.

It can only be partially verified that that the additive and functorial structures of Kitaev's fermionic proposal correspond to stacking phases and replacing symmetry groups, respectively, as the lattice model given in Ref. [71] was schematic.

A.8. Freed's fermionic proposal

We refer the reader to Refs. [38, 39] in view of the complexity of the proposal.

B. Field-Theoretic Argument for Weak-Index Interpretation

In this Appendix, we present a field-theoretic argument for Physical Result 7. To do so, we must first stipulate how to associate physical phases to cohomology classes (Apps. B.1 and B.2). Then we can check if the map α in Physical Result 7 on the mathematical side corresponds to the layering construction on the physical side (App. B.3).

The arguments below apply equally to the fermionic and the bosonic cases.

B.1. Kitaev's construction

We follow the prescription of Refs. [42, 71] to associate $(d-1)$ -dimensional SPT phases protected by on-site unitary symmetry G to cohomology classes $[c] \in h^{d-1}(BG)$. The construction is essentially a nonlinear sigma model with target space BG . There are some subtleties discussed in Refs. [42, 71] that we will sweep under the rug here.

To wit, we first associate to each map $c : BG \rightarrow F_{d-1}$ and spatial slice X the state

$$|\Psi(c, X)\rangle = \int_{\text{Map}(X, BG)} |m\rangle \otimes |\psi(c, m)\rangle \mathcal{D}m, \quad (63)$$

where m is a chiral field over X with target space BG , and $|\psi(c \circ m)\rangle$ is a pattern of SRE states that looks like $c(m(x)) \in F_{d-1}$ around $x \in X$. Then, to each cohomology class $[c] \in h^{d-1}(BG)$, we associate the $(d-1)$ -dimensional G -protected SPT phase represented by a system whose unique ground state on a spatial slice X is $|\Psi(c, X)\rangle$, where c is any representative of $[c]$.

B.2. A generalization to translational symmetry

We propose a generalization of the construction in Refs. [42, 71] that will enable us to associate d -dimensional SPT phases protected by discrete spatial translational symmetry \mathbb{Z} and on-site unitary symmetry G to cohomology classes $[c'] \in h^d(B(\mathbb{Z} \times G))$.

More specifically, over a spatial slice $Y = \mathbb{R} \times X$, where \mathbb{R} is the direction along which discrete spatial translational symmetry is assumed, we let there be two fields: a chiral field m' with target space BG and a background field $e^{i\phi}$ with target space $\mathbf{S}^1 \approx U(1)$ ¹⁵. The latter can be thought of as the vacuum expectation value of an order parameter characterizing the translational symmetry breaking. It should thus be constant over X and wind around \mathbf{S}^1 periodically along \mathbb{R} :

$$\phi(x_0 + 1) = \phi(x_0) + 2\pi, \quad (64)$$

which guarantees that $e^{i\phi(x_0+1)} = e^{i\phi(x_0)}$. Here, x_0 and x are the coordinates for \mathbb{R} and X , respectively. We have dropped x from the arguments of ϕ for brevity.

Now, we associate to each map $c' : \mathbf{S}^1 \times BG \rightarrow F_d$ and spatial slice $Y = \mathbb{R} \times X$ the state

$$|\Psi(c', \phi, X)\rangle = \int_{\text{Map}(Y, F_d)} |m'\rangle \otimes |\psi(c', \phi, m')\rangle \mathcal{D}m', \quad (65)$$

where $|\psi(c', \phi, m')\rangle$ is the pattern of SRE states that looks like $c'(e^{i\phi(x_0)}, m'(x_0, x))$ around $(x_0, x) \in \mathbb{R} \times X$. Then, to each cohomology class $[c'] \in h^d(B(\mathbb{Z} \times G)) \cong h^d(\mathbf{S}^1 \times BG)$, we associate the d -dimensional $(\mathbb{Z} \times G)$ -protected SPT phase represented by a system whose unique ground state on a spatial slice $Y = \mathbb{R} \times X$ is $|\Psi(c', \phi, X)\rangle$, where c' is any representative of $[c']$.

B.3. Weak-index interpretation

Take any $[c] \in h^{d-1}(BG)$ and let $[c'] \in h^d(B(\mathbb{Z} \times G))$ be its image under α . Since $F_{d-1} \simeq \Omega F_d$, the cohomology class $[c]$ can be represented by a map

$$c : BG \rightarrow \Omega F_d, \quad (66)$$

which sends each point of BG to a loop in F_d , or equivalently a map

$$c : \mathbf{S}^1 \times BG \rightarrow F_d \quad (67)$$

subject to the constraint that it sends all of $\{s_0\} \times BG$ to the basepoint of F_d , where s_0 denotes the basepoint of \mathbf{S}^1 . On the other hand, since $B(\mathbb{Z} \times G) \simeq \mathbf{S}^1 \times BG$, the cohomology class $[c']$ can also be represented by a map

$$c' : \mathbf{S}^1 \times BG \rightarrow F_d, \quad (68)$$

¹⁵We thank Ryan Thorngren for suggesting the idea of a background field.

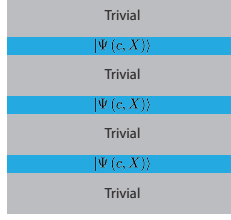


Figure 7: (color online). A stack of identical copies of $|\Psi(c, X)\rangle$ (blue) separated by trivial slabs (gray).

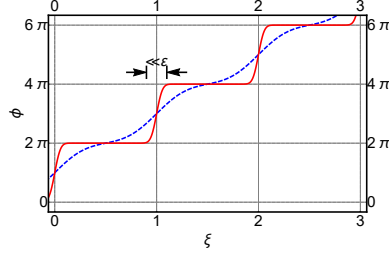


Figure 8: (color online). We deform ϕ from the dashed blue curve to the solid red curve, so that transitions occur within intervals of size much less than the short-distance cutoff ϵ for m' .

but without any constraint. One can show that α can be defined by setting

$$c' = c. \quad (69)$$

We now argue, by tinkering with the background field, that $|\Psi(c, \phi, X)\rangle$ can be obtained by stacking identical copies of $|\Psi(c, X)\rangle$ separated by trivial slabs (see Fig. 7). To that end, let us assume, in the spirit of Ref. [43], that there is a short distance cutoff ϵ for the chiral field m' . We deform ϕ according to Fig. 8: we create a series of plateaus and squeeze transitions between them to within a distance much less than ϵ from integral values of x_0 . Symmetry is preserved during the deformation, presumably so is the gap. Since the constant loop in F_d corresponds to a trivial $(d-1)$ -dimensional state, the $|\Psi(c, \phi, X)\rangle$ must now look trivial away from integral values of x_0 . This effectively decouples layers corresponding to different transitions between plateaus, each of which is nothing but a copy of $|\Psi(c, X)\rangle$. We have achieved the factorization

$$|\Psi(c, \phi, X)\rangle = \cdots \otimes |\Psi(c, X)\rangle \otimes |\text{trivial}\rangle \otimes |\Psi(c, X)\rangle \otimes |\text{trivial}\rangle \otimes \cdots. \quad (70)$$

C. Categorical Viewpoint

In this appendix, we revisit the Generalized Cohomology Hypothesis from a categorical perspective. As we will see, the Hypothesis can be stated more succinctly in categorical language (see App. F.2 for background).

C.1. Paraphrase of the Generalized Cohomology Hypothesis

The classification of SPT phases can be viewed as a sequence of contravariant functors

$$\mathcal{SPT}^d : \mathbf{Grp} \rightarrow \mathbf{Ab}^\delta \quad (71)$$

indexed by nonnegative integers $d \in \mathbb{N}$. Given a group G , $\mathcal{SPT}^d(G)$ is the discrete abelian group of d -dimensional G -protected SPT phases. Given a group homomorphism φ , $\mathcal{SPT}^d(\varphi)$ is the map defined by pulling back representations, as in Sec. 2.4.2. We can paraphrase the Generalized Cohomology Hypothesis as follows:

Generalized Cohomology Hypothesis (Categorical Version). There exists a generalized cohomology theory h such that there are natural isomorphisms

$$\mathcal{SPT}^d(G) \cong h^d(BG), \quad \forall d \in \mathbb{N}. \quad (72)$$

Note the left-hand side is defined physically while the right-hand side is purely mathematical. The Hypothesis bridges physics and mathematics.

But life is not always as good as natural isomorphisms. In practice, what one can do is to propose a *construction*, which can be viewed as a family of maps

$$h^d(BG) \rightarrow \mathcal{SPT}^d(G). \quad (73)$$

Such maps may or may not be bijective, but they had better be homomorphisms between discrete abelian groups and respect the functorial structure. In other words, they had better form a natural transformation for each d . Under certain conditions, this can be achieved through a redefinition of the additive or functorial structures of h^d if it is not already the case.

C.2. Further examples

Let us exemplify how this categorical lingo can be used.

We can say that Ref. [33] proposed a construction (at least for finite groups)

$$H^{d+2}(BG; \mathbb{Z}) \rightarrow \mathcal{SPT}^d(G), \quad (74)$$

and proved that the maps were well-defined. They actually form a natural transformation for each d as per App. D, though the original paper did not set out to prove this.

We can also say that Ref. [35] argued for the existence of natural isomorphisms (at least for finite groups)

$$\mathrm{Hom}(MSO_{d+1}(BG), U(1)) \rightarrow \mathcal{SPT}^d(G). \quad (75)$$

However, the paper did not give explicit formulas for these maps in the form of lattice models.

Suppose we can define maps

$$\mathrm{Hom}(MSO_{d+1}(BG), U(1)) \rightarrow \mathcal{SPT}^d(G) \quad (76)$$

that at least form a natural transformation for each d . Ref. [35] tried to elucidate the relationship between Ref. [33] and their proposal: “there exist SPT phases which appear to be nontrivial from the group cohomology point of view, but are trivial from the cobordism point of view,” and “there also exist SPT phases which are nontrivial from [the cobordism] point of view but are not captured by the group cohomology classification.” What was presumed in these remarks was a commutativity diagram for each d :

$$\begin{array}{ccc} H^{d+2}(BG; \mathbb{Z}) & \longrightarrow & \mathrm{Hom}(MSO_{d+1}(BG), U(1)) \\ \downarrow & \swarrow \text{dashed} & \\ \mathcal{SPT}^d(G) & & \end{array} \quad (77)$$

where the vertical arrow is the natural transformation (74), the dashed diagonal arrow is the natural transformation (76), and the horizontal arrow is a certain mathematically obvious natural transformation (assuming G is finite) that we do not intend to explain. The remarks quoted above amount to saying that the horizontal arrow does not have to be either injective or surjective.

Finally, we can say that what we did in App. B was to specify the horizontal arrows in the diagram below and argue that the diagram commutes:

$$\begin{array}{ccc} h^{d-1}(BG) & \longrightarrow & \mathcal{SPT}^d(G) \\ \downarrow \alpha & & \downarrow \text{layering} \\ h^d(B(\mathbb{Z} \times G)) & \longrightarrow & \mathcal{SPT}^d(\mathbb{Z} \times G) \end{array} \quad (78)$$

We used Kitaev’s construction [43] for the upper horizontal arrow and proposed a generalized construction for the lower horizontal arrow.

D. Additivity and Functoriality of the Group Cohomology Construction

In this subsection we will show, within the group cohomology construction [33] of bosonic SPT phases (for finite groups), that adding cohomology classes corresponds to stacking SPT phases (see Sec. 2.4.1), and that the map induced by a homomorphism between symmetry groups corresponds to replacing the symmetry group (see Sec. 2.4.2). We will begin with the 1-dimensional case.

D.1. 1-dimensional case

Let us review the construction in Ref. [33], specializing to 1 dimension. Take a finite symmetry group G . Consider a ring with N sites and associate to each site the $|G|$ -dimensional Hilbert space $\mathbb{C}G$, which has orthonormal basis $\{|g\rangle | g \in G\}$ and on which G acts according to $\rho_g |g_i\rangle = |gg_i\rangle$. We define $|\phi\rangle := \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle$ and $\hat{P}_i := \mathbb{I}^{\otimes(i-1)} \otimes |\phi\rangle \langle \phi| \otimes \mathbb{I}^{\otimes(N-i)}$. Then the Hamiltonian

$$\hat{H}(0) := - \sum_{i=1}^N \hat{P}_i \quad (79)$$

is local, preserves the symmetry, and has a unique, gapped ground state,

$$|\Psi(0)\rangle = |\phi\rangle^{\otimes N}. \quad (80)$$

Given a 2-cocycle $\nu \in \text{Hom}_G(G^3, U(1))$, we define a diagonal, local unitary operator,

$$\hat{U}(\nu) := \sum_{g_1, \dots, g_N \in G} \left[\nu(1, g_1, g_N)^{-1} \prod_{i=1}^{N-1} \nu(1, g_i, g_{i+1}) \right] |\{g_i\}\rangle \langle \{g_i\}|. \quad (81)$$

Then the Hamiltonian corresponding to ν is given by

$$\hat{H}(\nu) := \hat{U}(\nu) \hat{H}(0) \hat{U}(\nu)^\dagger, \quad (82)$$

which is local and symmetry-preserving because $\hat{H}(0)$ and $\hat{U}(\nu)$ are. It has a unique, gapped ground state,

$$|\Psi(\nu)\rangle = \frac{1}{\sqrt{|G|^N}} \sum_{g_1, \dots, g_N \in G} \left[\nu(1, g_1, g_N)^{-1} \prod_{i=1}^{N-1} \nu(1, g_i, g_{i+1}) \right] |g_1, \dots, g_N\rangle. \quad (83)$$

D.1.1. Adding cohomology classes = stacking SPT phases

Envision two rings as in App. D.1, corresponding to 2-cocycles ν and ν' , respectively. Stacking one ring on top of the other produces another 1-dimensional system. With an augmented Hilbert space $\mathbb{C}G \otimes \mathbb{C}G$ associated to each (composite) site, this composite system is no longer given by the group cohomology construction per se. It is, nevertheless, in the same phase as a system constructed as such, namely the one corresponding to the sum $\nu\nu'$ of ν and ν' , as we show below¹⁶. Thus, the mathematical addition of cocycles, and hence cohomology classes, corresponds precisely to the physical stacking of SPT phases.

To show that the composite system with the Hamiltonian

$$\hat{H}(\nu) \otimes \hat{H}(\nu') = \hat{U}(\nu) \hat{H}(0) \hat{U}(\nu)^\dagger \otimes \hat{U}(\nu') \hat{H}(0) \hat{U}(\nu')^\dagger \quad (84)$$

is in the same phase as the system with the Hamiltonian

$$\hat{H}(\nu\nu') = \hat{U}(\nu\nu') \hat{H}(0) \hat{U}(\nu\nu')^\dagger, \quad (85)$$

¹⁶Recall that there is an additive structure on the set of 2-cocycles, defined by $(\nu\nu')(g_0, g_1, g_2) := \nu(g_0, g_1, g_2)\nu(g_0, g_1, g_2)$. Addition of cocycles is written multiplicatively because, in physics, the composition law of $U(1)$ is usually considered multiplicative rather than additive.

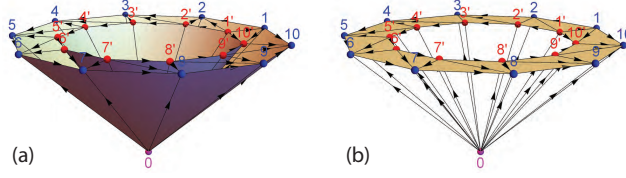


Figure 9: (color online). Two 1-dimensional systems, which consist of vertices labeled 1 through 10 (blue) and 1' through 10' (red), respectively, are stacked together to form a new 1-dimensional system. With the introduction of an auxiliary vertex 0 (magenta), a cone is formed for each system. The ground states $|\Psi(\nu)\rangle$ and $|\Psi(\nu')\rangle$ are then given by “integrating” ν and ν' over the two cones, respectively – this is a standard procedure in topological quantum field theories [89]. The coefficients in Eq. (86) and Eq. (88) are the “integrals” of ν' over the shaded “surfaces” (i.e. chains) in (a) and (b), respectively. The two are equal because the chains in (a) and (b) are homologous.

we first tensor the latter with a trivial ancillary ring, yielding $\hat{H}(\nu\nu') \otimes \hat{H}(0)$. Since $\hat{H}(\nu\nu') \otimes \hat{H}(0)$ is related to $\hat{H}(\nu) \otimes \hat{H}(\nu')$ by conjugation by the unitary operator

$$\begin{aligned} \hat{\tilde{U}}_1 &:= U(\nu\nu')U(\nu)^\dagger \otimes U(\nu')^\dagger \\ &= \sum_{\{g_i\}, \{g'_i\}} \left[\frac{\nu'(1, g'_1, g'_N)}{\nu'(1, g_1, g_N)} \prod_{i=1}^{N-1} \frac{\nu'(1, g_i, g_{i+1})}{\nu'(1, g'_i, g'_{i+1})} \right] |\{g_i\}\rangle \langle \{g'_i\}|, \end{aligned} \quad (86)$$

it suffices to find a path from \mathbb{I} to $\hat{\tilde{U}}_1$ via local unitary operators that preserve the symmetry. Here, $\{g_i\}$ and $\{g'_i\}$ are variables on the first and the second rings, respectively. By the cocycle condition $d\nu' = 0$, we have

$$\frac{\nu'(1, g_i, g_j)}{\nu'(1, g'_i, g'_j)} = \frac{\nu'(g'_i, g_i, g_j)}{\nu'(g'_i, g'_j, g_j)} \frac{\nu'(1, g'_j, g_j)}{\nu'(1, g'_i, g_i)} \quad (87)$$

for all i and j , which enables us to rewrite

$$\hat{\tilde{U}}_1 = \sum_{\{g_i\}, \{g'_i\}} \left[\frac{\nu'(g_1, g'_N, g_N)}{\nu'(g'_1, g_1, g'_N)} \prod_{i=1}^{N-1} \frac{\nu'(g'_i, g_i, g_{i+1})}{\nu'(g'_i, g'_{i+1}, g_{i+1})} \right] |\{g_i\}\rangle \langle \{g'_i\}|. \quad (88)$$

Geometrically, this amounts to replacing the chain shown in Fig. 9(a) by the chain shown in Fig. 9(b). In this new form, $\hat{\tilde{U}}_1$ would preserve the symmetry even if ν' failed to satisfy the cocycle condition. Take a path ν'_t in the space of 2-cochains that begins at the trivial 2-cochain and ends at ν' . Then

$$\hat{\tilde{U}}_t := \sum_{\{g_i\}, \{g'_i\}} \left[\frac{\nu'_t(g_1, g'_N, g_N)}{\nu'_t(g'_1, g_1, g'_N)} \prod_{i=1}^{N-1} \frac{\nu'_t(g'_i, g_i, g_{i+1})}{\nu'_t(g'_i, g'_{i+1}, g_{i+1})} \right] |\{g_i\}\rangle \langle \{g'_i\}|, \quad (89)$$

for $0 \leq t \leq 1$, is a path from \mathbb{I} to $\hat{\tilde{U}}_1$ via local unitary operators that preserve the symmetry, as desired.

D.1.2. Induced cohomology class = replaced symmetry group

Consider two possible symmetry groups G' and G and a homomorphism $\varphi : G' \rightarrow G$ between them. A 2-cocycle ν of G determines a 1-dimensional system representing a G -protected SPT phase via the construction in App. D.1. It has the Hilbert space $\mathbb{C}G$ associated to each site, the G -action $\rho_g |g_i\rangle = |gg_i\rangle$, and the Hamiltonian $\hat{H}(\nu)$. We denote this system by $(\mathbb{C}G, \rho, \hat{H}(\nu))$.

Precomposing ρ with φ , we obtain a G' -action on $\mathbb{C}G$:

$$(\rho \circ \varphi)_{g'} |g_i\rangle = \rho_{\varphi(g')} |g_i\rangle = |\varphi(g')g_i\rangle. \quad (90)$$

The Hamiltonian $\hat{H}(\nu)$ commutes with $\rho \circ \varphi$ since it does with ρ . Thus the same physical system can also be viewed as a representative of a G' -protected SPT phase. Physically, this amounts to forgetting

those symmetry operations in G that are not in the image of φ , and relabelling those in the image of φ by elements of G' in a possibly redundant manner. We denote this system by $(\mathbb{C}G, \rho \circ \varphi, \hat{H}(\nu))$.

On the other hand, the mathematical structure of group cohomology is such that every homomorphism $\varphi : G' \rightarrow G$ gives rise to an induced homomorphism φ^* from the discrete abelian group of 2-cocycles of G to the discrete abelian group of 2-cocycles of G' . More explicitly, this φ^* sends a 2-cocycle ν of G to the 2-cocycle

$$\begin{aligned} \varphi^* \nu : G' \times G' \times G' &\rightarrow U(1) \\ (g'_0, g'_1, g'_2) &\mapsto \nu(\varphi(g'_0), \varphi(g'_1), \varphi(g'_2)) \end{aligned} \quad (91)$$

of G' . For the given φ and ν , the 2-cocycle $\varphi^* \nu$ determines, via the construction in App. D.1, a system that represents a G' -protected SPT phase. We denote this system by $(\mathbb{C}G', \rho', \hat{H}(\varphi^* \nu))$.

A good construction of SPT phases should have functoriality built into its mathematical structure. It would therefore be ideal if the systems $(\mathbb{C}G', \rho', \hat{H}(\varphi^* \nu))$ and $(\mathbb{C}G, \rho \circ \varphi, \hat{H}(\nu))$ were actually the same, which is unfortunately false unless φ is an isomorphism. They are, however, in the same G' -protected SPT phase, as we now show.

To that end, let us recall that every group homomorphism can be factored as the composition of a surjective homomorphism and an inclusion. Thus it suffices to consider these two special cases.

First, suppose $\varphi : G' \rightarrow G$ is an inclusion. We will deform the system $(\mathbb{C}G, \rho \circ \varphi, \hat{H}(\nu))$ into the system $(\mathbb{C}G', \rho', \hat{H}(\varphi^* \nu))$ step by step. To begin, let S be a set of representatives of the right cosets of G' in G . That is,

$$G' s_1 \cap G' s_2 = \emptyset, \quad \forall s_1 \neq s_2 \in S, \quad (92)$$

$$\cup_{s \in S} G' s = G. \quad (93)$$

We can assume that the identity $1 \in G$ is contained in S . Given any $g \in G$, there is a unique pair $(g', s) \in G' \times S$ for which $g's = g$. We can thus rewrite every basis state $|g\rangle$ in the form $|g'\rangle \otimes |s\rangle$ and pretend that the Hilbert space $\mathbb{C}G$ is the tensor product of $\mathbb{C}G'$ and $\mathbb{C}S$. The G' -action $\rho \circ \varphi$ on $\mathbb{C}G$ then goes over into

$$(\rho \circ \varphi)_{g'}(|g'_i\rangle \otimes |s\rangle) = |g'g'_i\rangle \otimes |s\rangle. \quad (94)$$

Next, we choose a path \hat{W}_t of unitary operators on $\mathbb{C}G' \otimes \mathbb{C}S$ that acts trivially on $\mathbb{C}G'$ for all $t \in [0, 1]$, equals \mathbb{I} at $t = 0$, and sends $|g'_i\rangle \otimes \frac{1}{\sqrt{|S|}} \sum_{s \in S} |s\rangle$ to $|g'_i\rangle \otimes |1\rangle$ at $t = 1$. Since \hat{W}_t commutes with the G' -action for all t , so does the family of local unitary operators

$$\hat{U}_t := \hat{U}(\nu) \hat{W}_t^{\otimes N} \hat{U}(\nu)^\dagger. \quad (95)$$

The path $\hat{U}_t |\Psi(\nu)\rangle$ establishes an equivalence between $|\Psi(\nu)\rangle = \hat{U}_0 |\Psi(\nu)\rangle$ and

$$\begin{aligned} \hat{U}_1 |\Psi(\nu)\rangle &= \hat{U}(\nu) \hat{W}_1^{\otimes N} \hat{U}(\nu)^\dagger \hat{U}(\nu) |\Psi(0)\rangle \\ &= \hat{U}(\nu) \hat{W}_1^{\otimes N} |\Psi(0)\rangle \\ &= \hat{U}(\nu) \left(\hat{W}_1 \frac{1}{\sqrt{|G|}} \sum_{g' \in G'} |g'\rangle \otimes \sum_{s \in S} |s\rangle \right)^{\otimes N} \\ &= \hat{U}(\nu) \left(\frac{1}{\sqrt{|G'|}} \sum_{g' \in G'} |g'\rangle \otimes |1\rangle \right)^{\otimes N}. \end{aligned} \quad (96)$$

Restoring the old notation, the last expression reads

$$\hat{U}(\nu) \sum_{g_1, \dots, g_N \in G} \frac{\eta(\{g_i\})}{\sqrt{|G'|^N}} |\{g_i\}\rangle = \sum_{g_1, \dots, g_N \in G} \frac{\eta(\{g_i\})}{\sqrt{|G'|^N}} \left[\nu(1, g_1, g_N)^{-1} \prod_{i=1}^{N-1} \nu(1, g_i, g_{i+1}) \right] |\{g_i\}\rangle, \quad (97)$$

where $\eta(\{g_i\}) = 1$ if $g_i \in G'$ for all i and 0 otherwise. But this is related to the ground state

$$|\Psi(\varphi^* \nu)\rangle = \frac{1}{\sqrt{|G'|^N}} \sum_{g'_1, \dots, g'_N \in G'} \left[\nu(1, g'_1, g'_N)^{-1} \prod_{i=1}^{N-1} \nu(1, g'_i, g'_{i+1}) \right] |\{g'_i\}\rangle \quad (98)$$

of $(\mathbb{C}G', \rho', \hat{H}(\varphi^* \nu))$ by a symmetry-preserving isometry (the one induced by the inclusion $\mathbb{C}G' \subset \mathbb{C}G$), and hence equivalent to it.

Next, suppose $\varphi : G' \rightarrow G$ is a surjective homomorphism. We will deform the system $(\mathbb{C}G', \rho', \hat{H}(\varphi^* \nu))$ into the system $(\mathbb{C}G, \rho \circ \varphi, \hat{H}(\nu))$ step by step. To begin, let $R = \ker(\varphi)$, and T be a set of representatives of the left cosets of R in G' . That is,

$$t_1 R \cap t_2 R = \emptyset, \quad \forall t_1 \neq t_2 \in T, \quad (99)$$

$$\cup_{t \in T} tR = G'. \quad (100)$$

Given any $g' \in G'$, there is a unique pair $(t, r) \in T \times R$ for which $tr = g'$. We can thus rewrite every basis state $|g'\rangle$ in the form $|t\rangle \otimes |r\rangle$ and pretend that the Hilbert space $\mathbb{C}G'$ is the tensor product of $\mathbb{C}T$ and $\mathbb{C}R$. In this new form, the G' -action satisfies

$$\rho'_{g'}(|t\rangle \otimes |\phi_R\rangle) = |g'.t\rangle \otimes |\phi_R\rangle, \quad (101)$$

where $|\phi_R\rangle = \frac{1}{\sqrt{|R|}} \sum_{r \in R} |r\rangle$ and $g'.t$ is the unique element of T for which $\varphi(g'.t) = \varphi(g')\varphi(t)$. The ground state $|\Psi(\varphi^* \nu)\rangle$ of $(\mathbb{C}G', \rho', \hat{H}(\varphi^* \nu))$ goes over into

$$\begin{aligned} & \frac{1}{\sqrt{|G'|^N}} \sum_{\substack{t_1, \dots, t_N \in T \\ r_1, \dots, r_N \in R}} \left[(\varphi^* \nu)(1, t_1 r_1, t_N r_N)^{-1} \prod_{i=1}^{N-1} (\varphi^* \nu)(1, t_i r_i, t_{i+1} r_{i+1}) \right] |\{t_i\}\rangle \otimes |\{r_i\}\rangle \\ &= \frac{1}{\sqrt{|G'|^N}} \sum_{\substack{t_1, \dots, t_N \in T \\ r_1, \dots, r_N \in R}} \left[\nu(1, \varphi(t_1), \varphi(t_N))^{-1} \prod_{i=1}^{N-1} \nu(1, \varphi(t_i), \varphi(t_{i+1})) \right] |\{t_i\}\rangle \otimes |\{r_i\}\rangle \\ &= \left\{ \frac{1}{\sqrt{|T|^N}} \sum_{t_1, \dots, t_N \in T} \left[\nu(1, \varphi(t_1), \varphi(t_N))^{-1} \prod_{i=1}^{N-1} \nu(1, \varphi(t_i), \varphi(t_{i+1})) \right] |\{t_i\}\rangle \right\} \otimes |\phi_R\rangle^{\otimes N}. \quad (102) \end{aligned}$$

Removing the trivial ancilla $|\phi_R\rangle^{\otimes N}$, we obtain the equivalent state

$$\frac{1}{\sqrt{|T|^N}} \sum_{t_1, \dots, t_N \in T} \left[\nu(1, \varphi(t_1), \varphi(t_N))^{-1} \prod_{i=1}^{N-1} \nu(1, \varphi(t_i), \varphi(t_{i+1})) \right] |\{t_i\}\rangle. \quad (103)$$

Since φ gives a bijection between T and G , we can relabel the states $|t_i\rangle$ by elements of G , yielding

$$\begin{aligned} & \frac{1}{\sqrt{|T|^N}} \sum_{t_1, \dots, t_N \in T} \left[\nu(1, \varphi(t_1), \varphi(t_N))^{-1} \prod_{i=1}^{N-1} \nu(1, \varphi(t_i), \varphi(t_{i+1})) \right] |\{\varphi(t_i)\}\rangle \\ &= \frac{1}{\sqrt{|G|^N}} \sum_{g_1, \dots, g_N \in G} \left[\nu(1, g_1, g_N)^{-1} \prod_{i=1}^{N-1} \nu(1, g_i, g_{i+1}) \right] |\{g_i\}\rangle. \quad (104) \end{aligned}$$

This is nothing but the ground state $|\Psi(\nu)\rangle$ of $(\mathbb{C}G, \rho \circ \varphi, \hat{H}(\nu))$.

D.2. Higher-dimensional case

Take a finite symmetry group G . Consider a triangulated d -dimensional oriented closed manifold M together with a total ordering of the vertices¹⁷, which we accordingly label by $1, 2, \dots, N$. We denote by $\Delta_0, \dots, \Delta_d$ the vertices of a d -simplex Δ , with $\Delta_0 < \dots < \Delta_d$. The ordering $\Delta_0 < \dots < \Delta_d$ determines an orientation of Δ , which may or may not agree with that of M . We set $\mathcal{O}(\Delta) = 1$ if it does and $\mathcal{O}(\Delta) = -1$ otherwise. Given a $(d+1)$ -cocycle ν , the construction in Ref. [33] of $\hat{H}(\nu)$ and $|\Psi(\nu)\rangle$ is the same as in App. D.1 except that the unitary operator (81) should be replaced by

$$\hat{U}(\nu) := \sum_{\{g_i\}} \prod_{\Delta} \nu(1, g_{\Delta_0}, \dots, g_{\Delta_d})^{\mathcal{O}(\Delta)} |\{g_i\}\rangle \langle \{g_i\}|, \quad (105)$$

where Δ runs over the d -simplices of M .

D.2.1. Adding cohomology classes = stacking SPT phases

Take any d -cocycle ν' of G . Since $d\nu' = 0$, we have

$$\prod_{k=0}^d d\nu'(1, g'_{\Delta_0}, \dots, g'_{\Delta_k}, g_{\Delta_k}, \dots, g_{\Delta_d})^{(-1)^k} = 1 \quad (106)$$

for all $g'_{\Delta_0}, \dots, g'_{\Delta_d}, g_{\Delta_1}, \dots, g_{\Delta_d} \in G$. Expanding the left-hand side, one can show that

$$\prod_{\Delta} \left[\frac{\nu'(1, g_{\Delta_0}, \dots, g_{\Delta_d})}{\nu'(1, g'_{\Delta_0}, \dots, g'_{\Delta_d})} \right]^{\mathcal{O}(\Delta)} = \prod_{\Delta} \left[\prod_{k=0}^d \nu'(g'_0, \dots, g'_k, g_k, \dots, g_d)^{(-1)^k} \right]^{\mathcal{O}(\Delta)}. \quad (107)$$

The proof in App. D.1.1 can be immediately generalized to d dimensions by substituting Eq. (107) for Eq. (87), where the vertices of the composite system may be ordered either so that $1' < 1 < 2' < 2 < \dots < N' < N$ or so that $1' < \dots < N' < 1 < \dots < N$.

D.2.2. Induced cohomology class = replaced symmetry group

To generalize the proof in App. D.1.2 to d dimensions, one simply replaces all expressions of the form

$$\nu(1, g_1, g_N)^{-1} \prod_{i=1}^{N-1} \nu(1, g_i, g_{i+1}), \quad (108)$$

where ν is some 2-cocycle and $\{g_i\}$ is some indexed family of elements of either G' or G , by corresponding expressions of the form

$$\prod_{\Delta} \nu(1, g_{\Delta_0}, \dots, g_{\Delta_d})^{\mathcal{O}(\Delta)}, \quad (109)$$

where Δ runs over the d -simplices of M .

E. Proofs

In this appendix we collect proofs for mathematical results in this paper. We begin with some lemmas.

¹⁷Ref. [33] considered “branching structures” instead of total orderings of vertices, but this distinction is inconsequential.

E.1. Some lemmas

Lemma E.1. Let (F_n) be an Ω -spectrum and (X, x_0) be a pointed CW-complex. There is a natural split short exact sequence,

$$0 \longrightarrow \langle X, F_n \rangle \xrightarrow{i} [X, F_n] \xleftarrow[p]{s} [\{x_0\}, F_n] \longrightarrow 0 \quad (110)$$

with s induced by the projection $X \twoheadrightarrow \{x_0\}$, p induced by the inclusion $\{x_0\} \hookrightarrow X$, and i given by forgetting basepoints.

PROOF. The long exact sequence of reduced cohomology groups of the pair

$$((X \times I) / (X \times \partial I), (\{x_0\} \times I) / (\{x_0\} \times \partial I)) \quad (111)$$

breaks into short exact sequences, since there is an obvious retraction

$$(X \times I) / (X \times \partial I) \rightarrow (\{x_0\} \times I) / (\{x_0\} \times \partial I). \quad (112)$$

Now apply the suspension-loop adjunction and use the fact that $F_n \simeq \Omega F_{n+1}$. \square

Lemma E.2. Let (F_n) be an Ω -spectrum and (X, A) be a CW-pair with basepoint x_0 together with a retraction $\rho : X \twoheadrightarrow A$. There is a natural commutative diagram,

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \langle X/A, F_n \rangle & \xrightarrow{\tilde{\alpha}} & \langle X, F_n \rangle & \xrightarrow{\tilde{\beta}} & \langle A, F_n \rangle \longrightarrow 0 \\ & & \parallel & & \downarrow i & & \downarrow i \\ 0 & \longrightarrow & \langle X/A, F_n \rangle & \xrightarrow{\alpha} & [X, F_n] & \xrightarrow{\beta} & [A, F_n] \longrightarrow 0 \\ & & \downarrow & & \downarrow p & & \downarrow p \\ & & 0 & \longrightarrow & [\{x_0\}, F_n] & = & [\{x_0\}, F_n] \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \quad (113)$$

consisting of exact rows and columns, with $\tilde{\alpha}$ and α induced by the quotient map $X \twoheadrightarrow X/A$, $\tilde{\beta}$ and β induced by the inclusion $A \hookrightarrow X$, and i and p as in Lemma E.1. Furthermore, ρ induces splittings $\tilde{\sigma}$ and σ of the first and second rows, which fit into the commutative diagram

$$\begin{array}{ccc} \langle X, F_n \rangle & \xleftarrow{\tilde{\sigma}} & \langle A, F_n \rangle \\ i \downarrow & & \downarrow i \\ [X, F_n] & \xleftarrow{\sigma} & [A, F_n] \end{array} \quad (114)$$

PROOF. The exactness of the columns follows from Lemma E.1. The split exactness of the first row follows from the fact that the long exact sequence of reduced cohomology groups of (X, A) breaks into short exact sequences due to the existence of a retraction. The split exactness of the second row follows from diagram chasing. Commutativity and naturality are trivial to check. \square

Lemma E.3. Let (F_n) be an Ω -spectrum and X, Y be pointed CW-complexes. There exists an isomorphism,

$$\langle X \times Y, F_n \rangle \cong \langle X \vee Y \vee (X \wedge Y), F_n \rangle, \quad (115)$$

whose composition,

$$\tilde{\lambda} : \langle X \times Y, F_n \rangle \xrightarrow{\cong} \langle X, F_n \rangle \oplus \langle X \wedge Y, F_n \rangle \oplus \langle Y, F_n \rangle, \quad (116)$$

with the obvious isomorphism

$$\langle X \vee (X \wedge Y) \vee Y, F_n \rangle \cong \langle X, F_n \rangle \oplus \langle X \wedge Y, F_n \rangle \oplus \langle Y, F_n \rangle \quad (117)$$

is such that the canonical inclusions

$$\langle X, F_n \rangle \hookrightarrow \langle X \times Y, F_n \rangle, \quad (118)$$

$$\langle X \wedge Y, F_n \rangle \hookrightarrow \langle X \times Y, F_n \rangle, \quad (119)$$

$$\langle Y, F_n \rangle \hookrightarrow \langle X \times Y, F_n \rangle \quad (120)$$

are induced by the canonical projections $X \times Y \twoheadrightarrow X$, $X \times Y \twoheadrightarrow X \wedge Y$, and $X \times Y \twoheadrightarrow Y$, respectively, and that the canonical projections

$$\langle X \times Y, F_n \rangle \twoheadrightarrow \langle X, F_n \rangle, \quad (121)$$

$$\langle X \times Y, F_n \rangle \twoheadrightarrow \langle Y, F_n \rangle \quad (122)$$

are induced by the canonical inclusions $X \hookrightarrow X \times Y$ and $Y \hookrightarrow X \times Y$, respectively.

PROOF. Recall there is a stable splitting (Proposition 4I.1 of [78]),

$$\Sigma(X \times Y) \simeq \Sigma(X \vee (X \wedge Y) \vee Y). \quad (123)$$

Now apply the suspension-loop adjunction and use the fact that $F_n \simeq \Omega F_{n+1}$. The rest can be verified straightforwardly. \square

Lemma E.4. Let (F_n) be an Ω -spectrum and X, Y be pointed CW-complexes. There exists an isomorphism λ fitting into a natural commutative diagram

$$\begin{array}{ccc} \langle X \times Y, F_n \rangle & \xrightarrow{\tilde{\lambda}} & \langle X, F_n \rangle \oplus \langle X \wedge Y, F_n \rangle \oplus \langle Y, F_n \rangle \\ i \downarrow & & \downarrow \text{id} \oplus \text{id} \oplus i \\ [X \times Y, F_n] & \xrightarrow{\lambda} & \langle X, F_n \rangle \oplus \langle X \wedge Y, F_n \rangle \oplus [Y, F_n] \end{array} \quad (124)$$

where i is as in Lemma 5.1, $\tilde{\lambda}$ is as in Lemma E.3, and the canonical injection and projection

$$[Y, F_n] \hookrightarrow [X \times Y, F_n], \quad (125)$$

$$[X \times Y, F_n] \twoheadrightarrow [Y, F_n] \quad (126)$$

are induced by the canonical projection $X \times Y \twoheadrightarrow Y$ and injection $Y \hookrightarrow X \times Y$, respectively.

PROOF. Extend the columns into short exact sequences according to Lemma E.1. Then apply the Five Lemma to $\tilde{\lambda}^{-1}$ and the putative λ^{-1} . \square

E.2. Main proofs

PROOF OF LEMMA 5.1. Set $X = BG$ in Lemma E.1. \square

PROOF OF PROPOSITION 5.3. This is a special case of Proposition 5.9. \square

PROOF OF PROPOSITION 5.5. This is a special case of Proposition 5.9. \square

PROOF OF PROPOSITION 5.7. Set $X = BG_1$ and $Y = BG_2$ in Lemma E.4. \square

PROOF OF PROPOSITION 5.9. In Lemma E.2, set $X = B(G_1 \rtimes G_2)$, $A = BG_2$, and ρ to be induced by the canonical epimorphism $G_1 \rtimes G_2 \twoheadrightarrow G_2$. \square

PROOF OF LEMMA A.1. The desired natural transformation is the Bockstein homomorphism associated with the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0 \quad (127)$$

of abstract (i.e. without topology) abelian groups, which gives rise to a natural long exact sequence,

$$\cdots \rightarrow \tilde{H}^n(X; \mathbb{Z}) \rightarrow \tilde{H}^n(X; \mathbb{R}) \rightarrow \tilde{H}^n(X; U(1)) \rightarrow \tilde{H}^{n+1}(X; \mathbb{Z}) \rightarrow \tilde{H}^{n+1}(X; \mathbb{R}) \rightarrow \tilde{H}^{n+1}(X; U(1)) \rightarrow \cdots \quad (128)$$

The lemma will be established once we prove that

$$\tilde{H}^n(BG; \mathbb{R}) = 0 \quad (129)$$

for all $n \in \mathbb{Z}$ and finite groups G . By the universal coefficient theorem, this amounts to showing that $\text{Ext}^1(\tilde{H}_n(BG; \mathbb{Z}), \mathbb{R}) = \text{Hom}(\tilde{H}_n(BG; \mathbb{Z}), \mathbb{R}) = 0$. The Ext group is trivial because \mathbb{R} is a field. The Hom group is trivial because $\tilde{H}_n(BG; \mathbb{Z})$ is pure torsion, as per Remarks 3.6 and 3.7 and Corollary 5.4 in Chap. II of Ref. [77]. \square

F. Mathematical Background

F.1. Notions in algebraic topology

The definitions and constructions below are standard in algebraic topology. See e.g. Ref. [78] for detail.

Definition F.1 (pointed topological space). A pointed topological space (X, x_0) is a nonempty topological space X together with a privileged point $x_0 \in X$ called the basepoint. When the choice of x_0 is clear from the context, one may simply write X instead of (X, x_0) .

Recall from Sec. 2.2 that “map” always means continuous map.

Definition F.2 (pointed map). A pointed map between pointed topological spaces is a map that preserves basepoint.

Definition F.3 (topological group). A topological group is a topological space with a group structure such that both the multiplication and the inversion maps are continuous.

As in the main text (see Sec. 2.2), we will abbreviate “topological group” to simply “group” and assume that homomorphisms between topological groups are continuous.

Construction F.4. Given a topological space X , we can form the quotient space X/A from X by collapsing a subspace $A \subset X$. The image of A is the default basepoint of X/A .

Construction F.5. Given two pointed topological spaces (X, x_0) and (Y, y_0) , we define the wedge sum $X \vee Y$ to be $(X \sqcup Y) / \{x_0, y_0\}$. That is, it is formed from the disjoint union $X \sqcup Y$ by identifying x_0 and y_0 .

Construction F.6. Given two pointed topological spaces (X, x_0) and (Y, y_0) , we define the smash product $X \wedge Y$ to be $(X \times Y) / ((X \times \{y_0\}) \cup (\{x_0\} \times Y))$. It can be viewed as $(X \times Y) / (X \vee Y)$.

Construction F.7. Given a topological space X , we form the suspension SX from $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point.

Construction F.8. Given a pointed topological space (X, x_0) , we define the reduced suspension ΣX to be $(X \times I) / ((X \times \partial I) \cup (\{x_0\} \times I))$. Equivalently, it can be formed from SX by further collapsing $\{x_0\} \times I$. It can also be viewed as $\mathbf{S}^1 \wedge X$.

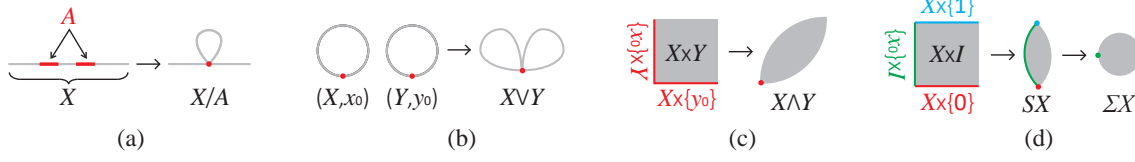


Figure 10: (color online). Illustration of the (a) quotient, (b) wedge sum, (c) smash product, (d) suspension, and reduced suspension constructions.

These constructions are illustrated in Fig. 10.

Definition F.9 (homotopy). A homotopy between two maps $f_0, f_1 : X \rightarrow Y$ is a map $f : X \times I \rightarrow Y$ such that

$$f(x, 0) = f_0(x), \quad f(x, 1) = f_1(x), \quad \forall x. \quad (130)$$

When such a map exists, f_0 and f_1 are said to be homotopic, and we write $f_0 \sim f_1$. This defines an equivalence relation, an equivalence class with respect to which is called a homotopy class. The set of homotopy classes of maps from X to Y is denoted by $[X, Y]$.

Definition F.10 (pointed homotopy). A pointed homotopy between two pointed maps $f_0, f_1 : (X, x_0) \rightarrow (Y, y_0)$ is a map $f : X \times I \rightarrow Y$ such that

$$f(x, 0) = f_0(x), \quad f(x, 1) = f_1(x), \quad \forall x, \quad (131)$$

$$f(x_0, t) = y_0, \quad \forall t. \quad (132)$$

When such a map exists, f_0 and f_1 are said to be homotopic in the pointed sense, and we write $f_0 \sim f_1$. This defines an equivalence relation, an equivalence class with respect to which is called a pointed homotopy class. The set of pointed homotopy classes of maps from (X, x_0) to (Y, y_0) is denoted by $\langle X, Y \rangle$.

Example F.11. The n -th homotopy group of a pointed topological space (Y, y_0) is $\pi_n(Y) := \langle \mathbf{S}^n, Y \rangle$. In particular, the fundamental group is $\pi_1(Y) := \langle \mathbf{S}^1, Y \rangle$, while the set of path components is $\pi_0(Y) := [\text{pt}, Y] \approx \langle \mathbf{S}^0, Y \rangle$.

Definition F.12 (homotopy equivalence). A homotopy equivalence between topological spaces X and Y is a pair of maps $f : X \rightarrow Y : g$ such that both $g \circ f$ and $f \circ g$ are homotopic to the identities. When such maps exist, X and Y are said to be homotopy equivalent, and we write $X \simeq Y$. This defines an equivalence relation, an equivalence class with respect to which is called a homotopy type.

Definition F.13 (pointed homotopy equivalence). A pointed homotopy equivalence between pointed topological spaces (X, x_0) and (Y, y_0) is a pair of pointed maps $f : (X, x_0) \rightarrow (Y, y_0) : g$ such that both $g \circ f$ and $f \circ g$ are homotopic to the identities in the pointed sense. When such maps exist, (X, x_0) and (Y, y_0) are said to be homotopy equivalent in the pointed sense, and we write $(X, x_0) \simeq (Y, y_0)$. This defines an equivalence relation, an equivalence class with respect to which is called a pointed homotopy type.

A single map $f : X \rightarrow Y$ or pointed map $f : (X, x_0) \rightarrow (Y, y_0)$ is sometimes said to be a homotopy equivalence or pointed homotopy equivalence, respectively, if a g with the above properties exists. Thus f is a homotopy equivalence or pointed homotopy equivalence if and only if it represents an invertible map in $[X, Y]$ or $\langle X, Y \rangle$, respectively. A homotopy equivalence or pointed homotopy equivalence is precisely an isomorphism in the homotopy category (see App. F.2).

Construction F.14. Given topological spaces X and Y , we can form the space $\text{Map}(X, Y)$ of maps from X to Y , endowed with the compact-open topology [78].

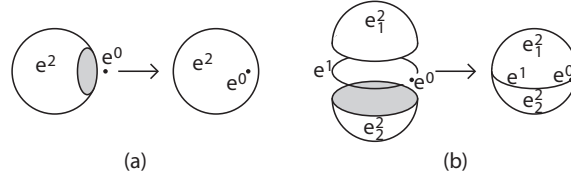


Figure 11: \mathbf{S}^2 can be constructed either (a) by attaching a single 2-cell e^2 to a single 0-cell e^0 , or (b) by attaching a single 1-cell e^1 (equator) to a single 0-cell e^0 and then attaching two 2-cells e_1^2 (northern hemisphere) and e_2^2 (southern hemisphere).

Construction F.15. Given pointed topological spaces (X, x_0) and (Y, y_0) , we can form the space $\text{Map}_*(X, Y)$ of pointed maps from (X, x_0) to (Y, y_0) , endowed with the compact-open topology.

Example F.16. Provided that X is sufficiently well-behaved (e.g. locally compact; see Proposition A.14 of Ref. [78]), a homotopy or pointed homotopy can alternatively be defined to be a path in the space $\text{Map}(X, Y)$ or $\text{Map}_*(X, Y)$, respectively. In this case, $[X, Y]$ and $\langle X, Y \rangle$ can be viewed as the sets of path components of $\text{Map}(X, Y)$ and $\text{Map}_*(X, Y)$, respectively.

Example F.17 (path space). The path space PY of a pointed topological space (Y, y_0) is defined to be the space $\text{Map}_*((I, 0), (Y, y_0))$. Intuitively, it is the space of paths in Y with y_0 as the initial point. There is a canonical map $PY \rightarrow Y$ sending a path p to its endpoint $p(1)$. The default basepoint of PY is the constant path.

Example F.18 (loop space). The loop space ΩY of a pointed topological space (Y, y_0) is defined to be the space $\text{Map}_*((\mathbf{S}^1, s_0), (Y, y_0))$. It can be viewed as the preimage of y_0 with respect to the map $PY \rightarrow Y$. Intuitively, it is the space of loops in Y based at y_0 . The default basepoint of ΩY is the constant loop.

Theorem F.19. The sequence $\Omega Y \rightarrow PY \rightarrow Y$, where the first map is the inclusion and the second map is as in Example F.17, is a fibration. It is called the path space fibration. \square

The definition of topological space is general enough to harbor wild examples. It is common in algebraic topology to work with better-behaved spaces, such as CW-complexes.

Construction F.20. Let us construct a topological space X inductively, as follows. Begin with a discrete topological space X^0 , called the 0-skeleton. For each $n \geq 1$, we form the n -skeleton X^n by “gluing” the boundaries of a family of n -disks to X^{n-1} along some maps $\varphi_\alpha : \partial \mathbf{D}^n \rightarrow X^{n-1}$. That is, we form the disjoint union $X^{n-1} \sqcup (\sqcup_\alpha \mathbf{D}_\alpha^n)$ and then identify $x \in \partial \mathbf{D}_\alpha^n$ with $\varphi_\alpha(x) \in X^{n-1}$ for all x and α . Finally, define $X = \cup_n X^n$ and declare a set in X to be open if and only if its intersections with all X^n ’s are open.

The homeomorphic image e_α^n of the interior of a \mathbf{D}_α^n is called an n -cell. A point in X^0 is called a 0-cell. Note that the φ_α ’s need not be injective.

Definition F.21 (CW-complex). A CW-complex is a topological space constructed as in Construction F.20, with the partition into cells retained as part of the data.

Example F.22. There are two common CW structures on \mathbf{S}^2 as illustrated in Fig. 11.

Example F.23. All closed manifolds of dimension $\neq 4$ can be given CW structures (the 4-dimensional case is an open question) [90, 91].

Definition F.24 (CW-group). A CW-group G is a CW-complex together with a topological group structure with the following properties [77, 92, 93]:

- (i) the inversion map sends n -cells to n -cells;
- (ii) $\forall g_1, g_2 \in G$ contained in some n_1 - and n_2 -cells respectively, $g_1 g_2$ is contained in a cell of dimension $\leq n_1 + n_2$.

These properties imply that the identity is a 0-cell.

Example F.25. All discrete groups can be viewed as CW-groups with each group element viewed as a 0-cell.

Example F.26. $O(n)$, $U(n)$, $Sp(n)$, and $SO(n)$ can all be given CW-group structures [94].

F.2. Categories, functors, and natural transformations

The definitions below are standard in category theory. See e.g. Ref. [95] for detail.

Definition F.27 (category). A category \mathcal{C} consists of

- (i) a class $\text{Obj}(\mathcal{C})$ of objects;
- (ii) a class $\text{Mor}(\mathcal{C})$ of morphisms (or arrows);
- (iii) a function $\text{dom} : \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$ called domain (or source) and a function $\text{cod} : \text{Mor}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$ called codomain (or target) – we denote by $\text{Hom}_{\mathcal{C}}(a, b)$ or simply $\text{Hom}(a, b)$, called the hom-class, the class of morphisms with domain a and codomain b , and use $f : a \rightarrow b$ to indicate that $\text{dom}(f) = a$ and $\text{cod}(f) = b$ –
- (iv) a function

$$\begin{aligned} \text{id} : \text{Obj}(\mathcal{C}) &\rightarrow \text{Mor}(\mathcal{C}) \\ a &\mapsto \text{id}_a \end{aligned} \tag{133}$$

called identity;

- (v) for each triple (a, b, c) of objects, a map

$$\begin{aligned} \text{Hom}(a, b) \times \text{Hom}(b, c) &\rightarrow \text{Hom}(a, c) \\ (f, g) &\mapsto g \circ f \text{ or } gf \end{aligned} \tag{134}$$

called composition – we say two morphisms f, g are composable if $g \circ f$ is defined –

such that the following axioms are satisfied:

1. associativity: $(h \circ g) \circ f = h \circ (g \circ f)$ for all composable morphisms f, g, h ;
2. identity: $\text{id}_a \in \text{Hom}(a, a)$ and $\text{id}_b \circ f = f \circ \text{id}_a = f$ for all objects a, b and morphisms $f \in \text{Hom}(a, b)$.

Example F.28. The category **Set** of sets has as objects the class of all sets, and as morphisms the class of all functions between sets. That is, $\text{Obj}(\mathbf{Set})$ consists of all sets, and given sets a, b , $\text{Hom}(a, b)$ consists of all functions from a to b . The composition is the usual composition of functions. Given a , id_a is the constant function on a .

Example F.29. The category **Top** of topological spaces has as objects all topological spaces, and as morphisms all maps between them.

Example F.30. The category \mathbf{Top}_* of pointed topological spaces has as objects all pointed topological spaces, and as morphisms all pointed maps between them.

Example F.31. The category \mathbf{Top}^2 of topological pairs has as objects all pairs (X, A) of topological spaces with $A \subset X$, and as $\mathrm{Hom}_{\mathbf{Top}^2}((X, A), (Y, B))$ all maps $f : X \rightarrow Y$ such that $f(A) \subset B$.

Example F.32. The category \mathbf{Grp} of groups has as objects all groups, and as morphisms all homomorphisms between them.

Example F.33. The category \mathbf{Ab}^δ of discrete abelian groups has as objects all discrete abelian groups, and as morphisms all homomorphisms between them.

Example F.34. The homotopy category \mathbf{Toph} of topological spaces has as objects all topological spaces, and $\mathrm{Hom}_{\mathbf{Toph}}(X, Y) := [X, Y]$.

Example F.35. The homotopy category \mathbf{Toph}_* of pointed topological spaces has as objects all pointed topological spaces, and $\mathrm{Hom}_{\mathbf{Toph}}(X, Y) := \langle X, Y \rangle$.

Definition F.36. A monomorphism, epimorphism, or isomorphism is a morphism that is left-cancellative, right-cancellative, or invertible (in the two-sided sense), respectively. Recall that f is called left- or right-cancellative if $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$ or $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$, respectively.

Example F.37. A monomorphism, epimorphism, or isomorphism in \mathbf{Set} is an injective, surjective, or bijective function, respectively.

Example F.38. A monomorphism, epimorphism, or isomorphism in \mathbf{Top} is an injective, surjective, or bijective map, respectively.

Example F.39. A monomorphism, epimorphism, or isomorphism in \mathbf{Grp} is an injective, surjective, or bijective homomorphism, respectively.

Example F.40. An isomorphism in \mathbf{Toph} or \mathbf{Toph}_* is a homotopy equivalence or pointed homotopy equivalence, respectively.

Definition F.41 (covariant functor). A covariant functor (or functor) \mathcal{F} from category \mathcal{C} to category \mathcal{D} , often written $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$, consists of

- (i) a function $\mathcal{F} : \mathrm{Obj}(\mathcal{C}) \rightarrow \mathrm{Obj}(\mathcal{D})$;
- (ii) a function $\mathcal{F} : \mathrm{Mor}(\mathcal{C}) \rightarrow \mathrm{Mor}(\mathcal{D})$;

such that the following axioms are satisfied:

- (i) \mathcal{F} maps $\mathrm{Hom}_{\mathcal{C}}(a, b)$ into $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(a), \mathcal{F}(b))$ for all $a, b \in \mathrm{Obj}(\mathcal{C})$;
- (ii) $\mathcal{F}(\mathrm{id}_a) = \mathrm{id}_{\mathcal{F}(a)}$ for all $a \in \mathrm{Obj}(\mathcal{C})$;
- (iii) $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all composable $f, g \in \mathrm{Mor}(\mathcal{C})$.

When \mathcal{F} is clear from the context, one often writes f_* instead of $\mathcal{F}(f)$.

Definition F.42 (contravariant functor). A contravariant functor (or cofunctor) \mathcal{F} from category \mathcal{C} to category \mathcal{D} , often written $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ (or $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{D}^{\mathrm{op}}$), consists of

1. a function $\mathcal{F} : \mathrm{Obj}(\mathcal{C}) \rightarrow \mathrm{Obj}(\mathcal{D})$;

2. a function $\mathcal{F} : \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$;

such that the following axioms are satisfied:

1. \mathcal{F} maps $\text{Hom}_{\mathcal{C}}(a, b)$ into $\text{Hom}_{\mathcal{D}}(\mathcal{F}(b), \mathcal{F}(a))$ for all $a, b \in \text{Obj}(\mathcal{C})$;
2. $\mathcal{F}(\text{id}_a) = \text{id}_{\mathcal{F}(a)}$ for all $a \in \text{Obj}(\mathcal{C})$;
3. $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all composable $f, g \in \text{Mor}(\mathcal{C})$.

When \mathcal{F} is clear from the context, one often writes f^* instead of $\mathcal{F}(f)$.

Example F.43. The forgetful functor $\mathcal{F}or : \mathbf{Top}_* \rightarrow \mathbf{Top}$ is a covariant functor that assigns to each pointed topological space (X, x_0) the topological space X with the basepoint forgotten, and to each pointed map $f : (X, x_0) \rightarrow (Y, y_0)$ the same f viewed as a map between unpointed topological spaces.

Example F.44. The loop space functor $\Omega : \mathbf{Top}_* \rightarrow \mathbf{Top}_*$ is a covariant functor that assigns to each $(X, x_0) \in \mathbf{Top}_*$ the loop space ΩX , and to each pointed map $f : (X, x_0) \rightarrow (Y, y_0)$ the map $\Omega f : \Omega X \rightarrow \Omega Y$ given by composition with f . That is, it sends a loop $l : (\mathbf{S}^1, s_0) \rightarrow (X, x_0)$ in (X, x_0) to the loop $f \circ l : (\mathbf{S}^1, s_0) \rightarrow (Y, y_0)$ in (Y, y_0) .

Example F.45. The classifying space functor $B : \mathbf{Grp} \rightarrow \mathbf{Top}_*$ is a covariant functor that assigns to each topological group G its classifying space BG , and to each homomorphism $\varphi : G' \rightarrow G$ a pointed map $\varphi_* : BG' \rightarrow BG$ (see App. F.4).

Definition F.46 (natural transformation). Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ be covariant functors. A natural transformation \mathcal{T} from \mathcal{F} to \mathcal{G} , often written $\mathcal{T} : \mathcal{F} \rightarrow \mathcal{G}$, is an assignment of a morphism $\mathcal{T}(a) : \mathcal{F}(a) \rightarrow \mathcal{G}(a)$ to each $a \in \text{Obj}(\mathcal{C})$ such that the following diagram commutes for all $a, b \in \text{Obj}(\mathcal{C})$ and all $f \in \text{Hom}_{\mathcal{C}}(a, b)$:

$$\begin{array}{ccc} \mathcal{F}(a) & \xrightarrow{\mathcal{T}(a)} & \mathcal{G}(a) \\ \mathcal{F}(f) \downarrow & & \downarrow \mathcal{G}(f) \\ \mathcal{F}(b) & \xrightarrow{\mathcal{T}(b)} & \mathcal{G}(b) \end{array} \quad (135)$$

A natural transformation between contravariant functors is defined the same way but with the vertical arrows in the diagram reversed.

Definition F.47 (natural isomorphism). A natural isomorphism \mathcal{T} is a natural transformation with all $\mathcal{T}(a)$ being isomorphisms.

F.3. Technical conventions

It is not only mathematically customary, but also physically justifiable, to work with “nice” categories of topological spaces and groups, because after all, pathological spaces and groups may be unphysical. Throughout the paper, apart from Apps. F.1-F.3, the following conventions shall be observed (adapted from Ref. [77]):

- (i) Unless a topological construction makes it impossible¹⁸, all topological spaces shall be CW-complexes, and the basepoints of all pointed topological spaces shall be 0-cells.
- (ii) All subspaces of CW-complexes shall be subcomplexes.

¹⁸For instance, the path or loop space of a pointed CW-complex may or may not be a pointed CW-complex. It is, however, always pointed homotopy equivalent to one [96].

- (iii) All groups shall be CW-groups.
- (iv) All subgroups shall be subcomplexes.
- (v) All binary products of topological spaces shall be compactly generated products.
- (vi) All objects in **Top**, **Top**_{*}, **Top**², **Toph**, **Toph**_{*}, and **Grp** shall be unpointed or pointed CW-complexes or CW-groups, as appropriate.

The CW approximation theorem implies that every topological space is weakly homotopy equivalent to a CW complex [78]. The following theorem (a generalization of Proposition 4.22 of Ref. [78]) then indicates that restricting to CW-complexes is hardly a loss of generality. It was also the reason why we were able to freely switch between homotopy equivalent spaces on numerous occasions in the main text.

Theorem F.48. Let $f : Y \rightarrow Z$ be a homotopy equivalence, or more generally weak homotopy equivalence, between topological spaces Y and Z . Then the induced maps

$$f_* : [X, Y] \rightarrow [X, Z], \quad (136)$$

$$f_* : \langle X, Y \rangle \rightarrow \langle X, Z \rangle, \quad (137)$$

$$f^* : [Z, X] \rightarrow [Y, X], \quad (138)$$

$$f^* : \langle Z, X \rangle \rightarrow \langle Y, X \rangle \quad (139)$$

are bijections for all CW-complexes X . □

F.4. Generalized cohomology theories

Definition F.49 (Eilenberg-Mac Lane space). Let G be a discrete group and n be a non-negative integer. If $n > 1$, we further require G to be abelian. A space X is called an Eilenberg-Mac Lane space $K(G, n)$ if

$$\pi_i(X) \cong \begin{cases} G, & i = n, \\ 0, & i \neq n, \end{cases} \quad (140)$$

for non-negative integers i . $K(G, n)$ exists and is unique up to homotopy equivalence. This allows us to abuse the terminology and speak of *the* Eilenberg-Mac Lane space $K(G, n)$.

Example F.50. $\mathbb{R}P^\infty$, \mathbb{Z} , \mathbb{S}^1 , and $\mathbb{C}P^\infty$ are $K(\mathbb{Z}_2, 1)$, $K(\mathbb{Z}, 0)$, $K(\mathbb{Z}, 1)$, and $K(\mathbb{Z}, 2)$, respectively.

Definition F.51 (classifying space). Let G be a group. A space BG is called a classifying space of G if there exists a principal G -bundle $\xi_G : EG \rightarrow BG$ satisfying either of the following equivalent conditions [77]:

- (i) Given any X , every principal G -bundle over X is isomorphic to the pull-back of ξ_G along a unique homotopy class of maps $f : X \rightarrow BG$.
- (ii) The map

$$\begin{aligned} [X, BG] &\rightarrow \left\{ \begin{array}{l} \text{isomorphism classes of prin-} \\ \text{cipal } G\text{-bundles over } X \end{array} \right\} \\ [f] &\mapsto [f^*(\xi_G)] \end{aligned} \quad (141)$$

is a bijection.

BG exists and is unique up to homotopy equivalence.

Table 4: Examples of classifying spaces. Recall that BG is unique only up to homotopy equivalence. Given here are the most widely used models for $\xi_G : EG \rightarrow BG$.

G	EG	BG	$\xi_G : EG \rightarrow BG$
\mathbb{Z}	\mathbb{R}	$\mathbf{S}^1 \approx U(1)$	$x \mapsto e^{i2\pi x}$
$U(1)$	$\mathbf{S}^\infty = \bigcup_{n=1}^\infty \mathbf{S}^{2n-1} \subset \bigcup_{n=1}^\infty \mathbb{C}^n$	$\mathbb{C}P^\infty = \bigcup_{n=0}^\infty \mathbb{C}P^n$	Identify $(z_1, \dots, z_n) \sim (z_1 e^{i\theta}, \dots, z_n e^{i\theta})$
\mathbb{Z}_2	$\mathbf{S}^\infty = \bigcup_{n=0}^\infty \mathbf{S}^n$	$\mathbb{R}P^\infty = \bigcup_{n=0}^\infty \mathbb{R}P^n$	Identify antipodes

Some simple examples of classifying spaces are given in Table 4. It turns out [77, 78] that

$$\pi_i(BG) = \pi_{i-1}(G). \quad (142)$$

Thus if G is a discrete group, then BG is a $K(G, 1)$. More generally, if a group G is a $K(G', n)$ as a topological space for some discrete G' , then BG is a $K(G', n+1)$. This is consistent with Example F.50 and Table 4.

Construction F.52 (explicit construction of classifying spaces). There is an explicit construction of $\xi : EG \rightarrow BG$ based on the usual geometric realization [77, 97]. It has the following properties:

- (i) Each EG is a CW-complex and each BG is a pointed CW-complex.
- (ii) $B : \mathbf{Grp} \rightarrow \mathbf{Top}_*$ is a covariant functor.
- (iii) $B(G_1 \times G_2)$ is homeomorphic to $BG_1 \times BG_2$.
- (iv) $B(G_1 \rtimes G_2)$ homotopy equivalent to $BG_1 \times_{G_2} EG_2$.
- (v) BG can be given an abelian group structure if G is abelian.

This will be our default model for BG .

The last property enables us to iterate the construction to produce B^2G, B^3G, \dots when G is an abelian group. If G is in addition discrete, then B^nG is a $K(G, n)$.

Definition F.53 (Ω -spectrum). An Ω -spectrum [75, 76, 78] is a family of pointed topological spaces indexed by integers,

$$\dots, F_{-2}, F_{-1}, F_0, F_1, F_2, \dots \quad (143)$$

together with pointed homotopy equivalences

$$F_n \xrightarrow{\simeq} \Omega F_{n+1} \quad (144)$$

for all n .

One can show that F_n determines all F_m 's with $m < n$ up to pointed homotopy equivalence. Moreover, shifting the index n turns an Ω -spectrum into another Ω -spectrum.

Example F.54 (Eilenberg-Mac Lane spectrum). Given any discrete abelian group A , the Eilenberg-Mac Lane spaces $K(A, n)$ form an Ω -spectrum, called the Eilenberg-Mac Lane spectrum of A [75, 76, 78]. More precisely, the Eilenberg-Mac Lane spectrum of A consists of

$$F_n = \begin{cases} K(A, n), & n \geq 0, \\ \text{pt}, & n < 0. \end{cases} \quad (145)$$

A generalized cohomology theory [75, 76] is a theory that satisfies the first six of the seven Eilenberg-Steenrod axioms [98, 99] plus Milnor's additivity axiom [100]. Inclusion of the seventh, dimension axiom of Eilenberg and Steenrod's would force the theory to be an ordinary one. Here we define generalized cohomology theories in an equivalent but more compact way [78].

Definition F.55 (reduced generalized cohomology theory). A reduced generalized cohomology theory consists of

- (i) a family of contravariant functors

$$\tilde{h}^n : \mathbf{Top}_* \rightarrow \mathbf{Ab}^\delta \quad (146)$$

indexed by integers n ;

- (ii) a natural transformation, called the coboundary map,

$$\delta : \tilde{h}^n(A) \rightarrow \tilde{h}^{n+1}(X/A) \quad (147)$$

for topological pairs (X, A) , for each n ;

such that the following axioms are satisfied:

- (i) homotopy: pointed homotopic maps in \mathbf{Top}_* induce identical homomorphisms in \mathbf{Ab}^δ ;
(ii) exactness: given any pair (X, A) , there is a long exact sequence

$$\cdots \xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q^*} \tilde{h}^{n+1}(X) \xrightarrow{i^*} \tilde{h}^{n+1}(A) \xrightarrow{\delta} \cdots \quad (148)$$

where $i : A \hookrightarrow X$ is the inclusion map and $q : X \twoheadrightarrow X/A$ is the quotient map;

- (iii) wedge: given any family of pointed spaces, (X_α) , the inclusion maps $X_\alpha \hookrightarrow \vee_\alpha X_\alpha$ induce an isomorphism

$$\tilde{h}^n(\vee_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha \tilde{h}^n(X_\alpha). \quad (149)$$

Definition F.56 (unreduced generalized cohomology theory). An (unreduced) generalized cohomology theory consists of

- (i) a family of contravariant functors

$$h^n : \mathbf{Top}^2 \rightarrow \mathbf{Ab}^\delta \quad (150)$$

indexed by integers n ;

- (ii) a natural transformation, called the coboundary map,

$$\delta : h^n(A, \emptyset) \rightarrow h^{n+1}(X, A) \quad (151)$$

for topological pairs (X, A) , for each n ;

such that the following axioms are satisfied:

- (i) homotopy: homotopic maps in \mathbf{Top}^2 induce identical homomorphisms in \mathbf{Ab}^δ ;
(ii) exactness: given any pair (X, A) , there is a long exact sequence

$$\cdots \xrightarrow{\delta} h^n(X, A) \xrightarrow{j^*} h^n(X, \emptyset) \xrightarrow{i^*} h^n(A, \emptyset) \xrightarrow{\delta} h^{n+1}(X, A) \xrightarrow{j^*} h^{n+1}(X, \emptyset) \xrightarrow{i^*} h^{n+1}(A, \emptyset) \xrightarrow{\delta} \cdots \quad (152)$$

where $i : (A, \emptyset) \rightarrow (X, \emptyset)$ and $j : (X, \emptyset) \rightarrow (X, A)$ are the inclusion maps.

- (iii) excision: given a triple (X, A, B) with $B \subset A \subset X$, the quotient map $(X, A) \rightarrow (X/B, A/B)$ induces an isomorphism

$$h^n(X/B, A/B) \xrightarrow{\cong} h^n(X, A); \quad (153)$$

- (iv) additivity: given any family of pairs, (X_α, A_α) , the inclusion maps $(X_\alpha, A_\alpha) \rightarrow (\sqcup_\alpha X_\alpha, \sqcup_\alpha A_\alpha)$ induce an isomorphism

$$h^n(\sqcup_\alpha X_\alpha, \sqcup_\alpha A_\alpha) \xrightarrow{\cong} \prod_\alpha h^n(X_\alpha, A_\alpha). \quad (154)$$

Every reduced generalized cohomology theory canonically determines an unreduced generalized cohomology theory, and vice versa, as follows. Given a reduced theory \tilde{h} , we define an unreduced theory h according to

$$h^n(X, A) := \tilde{h}^n(X/A), \quad (155)$$

with the convention $X/\emptyset := X \sqcup \text{pt}$. Given an unreduced theory h , we define a reduced theory \tilde{h} according to

$$\tilde{h}^n(X) := h^n(X, \text{pt}). \quad (156)$$

To make contact with Definitions 3.1 and 3.2, we need the pivotal Brown representability theorem (see e.g. Ref. [101] or Theorems 4.58 and 4E.1 of Ref. [78]).

Theorem F.57 (Brown representability theorem). Every Ω -spectrum $(F_n)_{n \in \mathbb{Z}}$ defines a reduced generalized cohomology theory \tilde{h} according to

$$\tilde{h}^n(X) := \langle X, F_n \rangle. \quad (157)$$

Conversely, every reduced generalized cohomology theory can be represented by an Ω -spectrum this way. \square

Definitions 3.1 and 3.2 differ from Definitions F.56 and F.55 in two subtle ways, even when the Brown representability theorem is assumed. First, Definitions 3.1 and 3.2 treated Ω -spectrum as part of the data of a generalized cohomology theory, but in reality different Ω -spectra can represent the same theory (although, in the category of spectra, a representing spectrum is determined by the theory up to isomorphism, in view of the Yoneda lemma). It was because of the physical interpretations of Ω -spectrum that we decided to treat it as part of the data. Second, in Definition 3.1, an unreduced generalized cohomology theory was only evaluated on individual spaces not pairs. The connection is given by

$$h^n(X) := h^n(X, \emptyset). \quad (158)$$

It is then easy to show that

$$h^n(X) \cong [X, F_n] \quad (159)$$

for any Ω -spectrum (F_n) that represents the corresponding reduced theory \tilde{h} , in accord with Definition 3.1.

Table 2 contains some classic generalized cohomology theories alongside with Ω -spectra that represent them.

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